Divisors of positive integers

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In mathematics, a theorem is a powerful thing because it can be proved. In these pages, we will do something more like make observations and develop inferences. We curve fit to data tables and let our computer do calculations on whole numbers. We hope you enjoy the ride.

There is a theorem, called the fundamental theorem of arithmetic, it is part of the foundations of mathematics. We will start with the fundamental theorem of arithmetic. This theorem applies to positive integers, those are whole numbers one and greater. Rational numbers and other non-integer real numbers will not be considered here. In short, every positive integer has a unique (up to reordering) factorization in prime numbers. This is part of number theory. Let b be an arbitrary chosen positive integer. In words, the fundamental theorem of arithmetic states that every positive integer has a unique prime factorization. Now prime numbers are only divisible by one and themselves. The set of all prime numbers goes like this –

$\mathbf{P} = \{2, 3, 5, 7, 11, \ldots\}$ (expression 1)

The fundamental theorem of arithmetic goes like this

$b = p0^{e0}p1^{e1}p2^{e2} \ldots pd^{ed}$. (expression 2)

where the pi are prime numbers, and the ej are positive integer exponents. So for every positive integer b, there is a unique finite set of p0, p1, … and e0, e1, … that describe b. In other words, the 'pi's are the d prime factors of order ei. Simply, b has a unique prime factorization.

For example $18 = 2*3^2$.

And 2*3*3 is the only prime factorization for 18.

Here is another example 30 = 2*3*5.

Similarly, 2*3*5 is the unique prime factorization for 15.

And if we choose to put the prime numbers from smallest to largest, and not put any one as factors, then there is only one unique of factorization for every integer. This is standard mathematics I learned in school.

More Preliminaries

Now there is a concept of discrete divisors. By definition, the list of discrete divisors of a positive integer b includes 1. The greatest discrete divisor of a positive integer b is less than or equal to b.

General Case

Let d = $p1^{e1}*p2^{e2}*p3^{e3}*$ … $*pn^{en}$.

Require d to be a positive integer. And then d has a unique prime factorization.

For d, there are n prime factors. The prime factors are p1, p2, … pn.

The total number of divisors of d is (e1+1)*(e2+1)*(e3+1)*…*(en+1).

Since a list of proper divisors of a number does not include that number itself, we have the total number of proper divisors of d is (e1+1)*(e2+1)*(e3+1)*…*(en+1) -1. We call this (expression 3).

This previous paragraph is a big idea involving combinatorics. Suppose you have one cup with 4 balls and a second cup with 3 balls. All the balls are a different color. How many different ball pairs are possible? The answer is 3*4 = 12. There are 4 choices from the first cup and 3 choices from the second cup. Hopefully this paragraph has shed some light on the subject of combinatorics.

Computer Coding

Using a wonderful computer algebra system called Maple, I was able to do some simple computer coding to make a Discrete_Divisors(b) procedure. I worked many examples to try to get a handle on the behavior of this Discrete_Divisors(b) procedure

Now assume b is a prime, or b is a semi-prime, that is b = p*q where p and q are both prime numbers. If b is a prime number then the count of its Discrete Divisors is 1 and its only discrete divisor is 1. This is inferred by observation and is not a theorem from my point of view.

Similarly,

Also, if b is the product of d distinct primes, each with exponent 1, then the count of the discrete divisors will be d. In other words, assume

b = p1*p2* … * pd. (expression 4)

Then b has exactly d discrete divisors. Again, by observation of computer calculation.

Making a curve fit from observed data from expression 3, in our example we used consecutive primes starting with 3. We see that Count_Discrete_Divisors(3*5*7*11*13) = 31 = 2^5 -1. These are called Mersenne numbers and have the form M(n) = 2^n -1. So our observed generalization is that

If b = p1*p2* … * pd and all p1, p2, … pd are distinct primes, then

Count_Discrete_Divisors(b) = 2^b -1. (expression 5)

Next form of b.

Assume b has one repeated prime factor. (repeated ‘a’ times) So

b = $p0^a$.

written with less pretty typesetting, we have

b = p0^a. (expression 6)

For example, choose p0=3 and let a = 1,2,3, … c.

Here is our data table

‘a’ count_of_discrete_divisors(3^a).

1	1
2	2
3	3
4	4

Numerical evidence shows that a number that is a single prime power with exponent ‘a’, will have exactly ‘a’ different discrete divisors. That is, our number b has one repeated prime factor only. This is a nice bit of knowledge that allows us to go on to our next section.

Section 2 – 2 repeated prime factors.

Suppose our number b now has 2 different primes in its factorization, and those primes are repeated an arbitrary number of times.

For example, let b = 3^4*11^6.

More generally, now b = $p0^{e0}*p1^{e1}$. Where p0 < p1 and p0 and p1 are prime numbers and e0 and e1 are non-negative integrs.

Now if e1 = 0, then we have $p1^{e1}$ = 1, and this is a degenerate case which simplifies to what we just did.

Summary of more Maple calculation

proper divisor count for (two prime with repition) factorization

b = (p0^e0)*(p1^e1)

e0\e1	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	3				
2	2	5				
3	3	7				
4	4	9				
5	5	11				
6	6	13				

note for later, Mersenne numbers Me = 2^e - 1.

Another note, to move down columns in this 'Cayley' table

recursive_1 r(0) = 0 or something else and r(n) = 2*r(n-1) + 1.

It looks like this second column has a pattern, after the first two data points.

for e0 > -1, we have count_divisors(3^e0 * 5) = 2*e0 + 1.

Where 5 was chosen as an arbitrary prime.

More generally, count_divisors((p0^e0) *p1) = 2*e0 + 1.

Now for the third column of our table. Our numbers b now have

the form b=p0^e0 * p1^e1 * p2^e2.

We choose p0 = 3, p1 = 11 and p2 = 13 for our examples.

To start, let e1 = e2 = 1.

Raw computer data yields

Proper_Divisor_count(3*11*13) = 7

Proper_Divisor_count(3^2*11*13) = 11

Proper_Divisor_count(3^3*11*13) = 15

i'm seeing a pattern here

Proper_Divisor_count(3^4*11*13) = 19

general expression

Proper_Divisor_count(3^d*11*13) = 4*d + 3. (expression 7)

That checks out.

Appendix 1

Divisors Procedure in Maple

```
>

> Divisors := proc (n) local d, count; description " Enumerate all proper divisors.  Assume a positive integer input.     Then count the proper divisors."; print(" Input is ", n, " Begin calculation."); count := 0; for d to (1/2)*n do if `mod`(n, d) = 0 then count := count+1; print("   One proper divisor of ", n, " is the number ", d) end if end do; print(" and that is all of them.  "); print(" count of proper divisors is ", count) end proc;


> Describe(Divisors);

%;


#  Enumerate all proper divisors. Assume a positive integer input. Then count

# the proper divisors.

Divisors( n )


# example for Proper_Divisors(4)

> Divisors(2^2);

                 " Input is ", 4, " Begin calculation."

           "   One proper divisor of ", 4, " is the number ", 1

           "   One proper divisor of ", 4, " is the number ", 2

                   " and that is all of them.  "

                  " count of proper divisors is ", 2

# good fun
```

End Appendix 1

Appendix 2

Examples of Discrete Prime Divisors procedure to demonstrate that square free positive integers with exactly d prime divisors have exactly d discrete prime divisors. In other words, assume

b = p1*p2* … * pd.

See examples ~

Proper divisors of a prime number has only one as its prime divisor.

> ProperDivisors(19);

" Input is ", 19, " Begin calculation."

" One proper divisor of ", 19, " is the number ", 1

" and that is all of them. "

" count of proper divisors is ", 1

> ProperDivisors(17*(11*(5*7)*13));

so 85085 = 5*7*11*13*17 has 5 distinct prime divisors and is square free. (prime divisors are without repetition.)

" Input is ", 85085, " Begin calculation."

" One proper divisor of ", 85085, " is the number ", 1

" One proper divisor of ", 85085, " is the number ", 5

" One proper divisor of ", 85085, " is the number ", 7

" One proper divisor of ", 85085, " is the number ", 11

" One proper divisor of ", 85085, " is the number ", 13

" One proper divisor of ", 85085, " is the number ", 17

" One proper divisor of ", 85085, " is the number ", 35

" One proper divisor of ", 85085, " is the number ", 55

" One proper divisor of ", 85085, " is the number ", 65

" One proper divisor of ", 85085, " is the number ", 77

" One proper divisor of ", 85085, " is the number ", 85

" One proper divisor of ", 85085, " is the number ", 91

" One proper divisor of ", 85085, " is the number ", 119

" One proper divisor of ", 85085, " is the number ", 143

" One proper divisor of ", 85085, " is the number ", 187

" One proper divisor of ", 85085, " is the number ", 221

" One proper divisor of ", 85085, " is the number ", 385

" One proper divisor of ", 85085, " is the number ", 455

" One proper divisor of ", 85085, " is the number ", 595

" One proper divisor of ", 85085, " is the number ", 715

" One proper divisor of ", 85085, " is the number ", 935

" One proper divisor of ", 85085, " is the number ", 1001

" One proper divisor of ", 85085, " is the number ", 1105

" One proper divisor of ", 85085, " is the number ", 1309

" One proper divisor of ", 85085, " is the number ", 1547

" One proper divisor of ", 85085, " is the number ", 2431

" One proper divisor of ", 85085, " is the number ", 5005

" One proper divisor of ", 85085, " is the number ", 6545

" One proper divisor of ", 85085, " is the number ", 7735

" One proper divisor of ", 85085, " is the number ", 12155

" One proper divisor of ", 85085, " is the number ", 17017

" and that is all of them. "

" count of proper divisors is ", 31

note that $31 = 2^5 - 1$.

> ProperDivisors(5*7);

" Input is ", 35, " Begin calculation."

"   One proper divisor of ", 35, " is the number ", 1

"   One proper divisor of ", 35, " is the number ", 5

"   One proper divisor of ", 35, " is the number ", 7

" and that is all of them. "

" count of proper divisors is ", 3

so 5*7 = 35 is a semi prime and has 2 distinct prime divisors.

our relationship of note is with Mersenne numbers 3 = $2^2 – 1$

another example to hammer this relationship home.

b = 11*13*19.

> ProperDivisors(19*(11*13));

" Input is ", 2717, " Begin calculation."

"   One proper divisor of ", 2717, " is the number ", 1

"   One proper divisor of ", 2717, " is the number ", 11

"   One proper divisor of ", 2717, " is the number ", 13

"   One proper divisor of ", 2717, " is the number ", 19

"   One proper divisor of ", 2717, " is the number ", 143

"   One proper divisor of ", 2717, " is the number ", 209

"   One proper divisor of ", 2717, " is the number ", 247

" and that is all of them. "

" count of proper divisors is ", 7

we see that our count is 7 and $2^3 – 1$. And our number of primes in b is 3.

woo hoo. Insight by Matt C Anderson.

and just because appendix 2 is not long enough, we will do an example of 4 consecutive primes starting at 11, and find a count of 15 discrete divisors. Since 2^4 – 1 is 15 and those are Mersenne numbers.

b = 11*13*17*19

```
> ProperDivisors(17*(11*13)*19);

print(`output redirected...`); # input placeholder

                " Input is ", 46189, " Begin calculation."

          "   One proper divisor of ", 46189, " is the number ", 1

          "   One proper divisor of ", 46189, " is the number ", 11

          "   One proper divisor of ", 46189, " is the number ", 13

          "   One proper divisor of ", 46189, " is the number ", 17

          "   One proper divisor of ", 46189, " is the number ", 19

         "   One proper divisor of ", 46189, " is the number ", 143

         "   One proper divisor of ", 46189, " is the number ", 187

         "   One proper divisor of ", 46189, " is the number ", 209

         "   One proper divisor of ", 46189, " is the number ", 221

         "   One proper divisor of ", 46189, " is the number ", 247

         "   One proper divisor of ", 46189, " is the number ", 323

         "   One proper divisor of ", 46189, " is the number ", 2431

         "   One proper divisor of ", 46189, " is the number ", 2717

         "   One proper divisor of ", 46189, " is the number ", 3553

         "   One proper divisor of ", 46189, " is the number ", 4199

                    " and that is all of them.  "

                  " count of proper divisors is ", 15
```

fun to observe.

there it is.

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