A prime number "game of life"

Martin Raab

In this paper, the author discusses the existence of a real number y such that $q = \lfloor p\#*y \rfloor$ is a prime number for every $p \geq 2$.

Definitions:
$\lfloor x \rfloor$ is the floor function, i.e. the largest integer less than x.
p# is the primorial function, the product of all primes $\leq p$. $1\# = 1$ by convention.
[a, b[is the open interval containing real numbers x such that $a \leq x < b$.

Objective? There is no objective.

The objective of most any kind of game is to get acquainted with the cornucopia of different situations in everyday life. In the same fashion, playing with numbers can clear the path to surprising mathematical insights one didn't originally intend to find.
While large portions of this paper consist of artisan work with known tools of number theory, there may something to be gained in-between the lines, for example a way to calculate $\theta(x)$ directly from known values of $Li(x)-\pi(x)$ which I haven't seen before in literature. There are also a couple of open problems which may or may not be worth investigating further.

Setting the stage

Any real number y such that $q = \lfloor p\#*y \rfloor$ is a prime number for every $p \geq 2$ must lie in the interval [1, 1.5[. If $y < 1$, then $\lfloor 2\#*y \rfloor < 2$, which is less than the first prime number. For $y \geq 1.5$, $\lfloor 2\#*y \rfloor$ must be an odd prime $2m-1$ for some integer $m > 1$. Then $\lfloor 3\#*y \rfloor$ is a number in the interval $[6m-3, 6m-1]$, only the latter of which can be prime because $6m-3$ is divisible by 3 and $6m-2$ is divisible by 2. Likewise, any integer in the successive intervals $[p\#*m-p, p\#*m-1]$ is divisible by a small prime $\leq p$ except for $p\#*m-1$. Hence there would have to be an integer m for which $p\#*m-1$ is always prime, i.e. the sequence $f(0) = m-1$, $f(s+1) = (f(s)+1)*p_s-1$ with p_s being the s^{th} prime number must only produce primes, which is practically impossible (but hasn't been disproven yet!).
It is possible, however, to find arbitrarily long sequences for some m, albeit only finite ones:

r	m	r	m
5	2	23	384,427
7	2	29	16,114,470
11	9	31	259,959,472
13	9	37	13,543,584,514
17	9	41	100,318,016,379
19	224,719	43	100,318,016,379

Table 1. Smallest integers m for which $p\#*m-1$ is prime for $p \leq r$. A quick-and-dirty analysis suggests that the next entry in this table is not likely to occur before 10^{14}.

To find out about an admissible value for y, we have to look for primes q, where each q requires the primality of $\lfloor q/p\#*r\#\rfloor$ for every $r \le p$. In the following setup, Y will denote the interval in which any number y as described above can lie, setting Y = [1, 1.5[for a start, and then successively sharpen the bounds where, for every p, Y*p# must contain at least one prime to satisfy the conditions.

- Stage 1: p = 2 → 2*Y = [2, 3[→ ⌊2*Y⌋ = 2 – no further action required yet
- Stage 2: p = 3 → 6*Y = [6, 9[– 7 is the only prime in here, so Y narrows down to [$^7/_6$, $^8/_6$[
- Stage 3: p = 5 → 30*Y = [35, 40[– one prime (37) in this interval, Y adjusts to [$^{37}/_{30}$, $^{38}/_{30}$[
- Stage 4: p = 7 → 210*Y = [259, 266[– once again, one prime in here and any possible y lies between $^{263}/_{210}$ and $^{264}/_{210}$
- Stage 5: p = 11 → 2,310*Y = [2,893, 2,904[– now it's getting interesting, since there are two primes in that interval, 2,897 and 2,903. We will split Y into two separate intervals, Y_1 = [$^{2,897}/_{2,310}$, $^{2,898}/_{2,310}$[and Y_2 = [$^{2,903}/_{2,310}$, $^{2,904}/_{2,310}$[and proceed
- Stage 6: p = 13 → each Y_1 and Y_2 brings forth one prime, 37,663 and 37,747 respectively. Cutting back both Y's, then
- Stage 7: p = 17 → Y_1 leads to 640,279, whereas Y_2 gives 641,701 and 641,713, and Y_2 will be split into Y_2 and Y_3 in accordance with the primes
- Stage 8: p = 19 → 9,699,690*Y_1 contains a prime triplet, 12,165,311+d for d={0, 2, 6}, while Y_2 and Y_3 both fail to produce any further prime
- etc.

The results can be put into a nice grid:

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<tr><td>⌊Y⌋</td><td colspan="11">1</td></tr>
<tr><td>* 2</td><td colspan="11">+ 0</td></tr>
<tr><td>* 3</td><td colspan="11">+ 1</td></tr>
<tr><td>* 5</td><td colspan="11">+ 2</td></tr>
<tr><td>* 7</td><td colspan="11">+ 4</td></tr>
<tr><td>* 11</td><td colspan="9">+ 4</td><td colspan="2">+ 10</td></tr>
<tr><td>* 13</td><td colspan="9">+ 2</td><td colspan="2">+ 8</td></tr>
<tr><td>* 17</td><td colspan="9">+ 8</td><td>+ 2</td><td>+ 14</td></tr>
<tr><td>* 19</td><td>+ 10</td><td colspan="6">+ 12</td><td colspan="2">+ 16</td><td>(end)</td><td>(end)</td></tr>
<tr><td>* 23</td><td>+ 16</td><td colspan="6">+ 8</td><td>+ 16</td><td>+ 18</td><td rowspan="4"></td><td rowspan="4"></td></tr>
<tr><td>* 29</td><td>+ 16</td><td colspan="2">+ 6</td><td>+ 14</td><td colspan="2">+ 24</td><td>+ 26</td><td>+ 10</td><td>(end)</td></tr>
<tr><td>* 31</td><td>(end)</td><td>+ 14</td><td>+ 24</td><td>(end)</td><td>+ 6</td><td>+ 14</td><td>(end)</td><td>(end)</td><td rowspan="2"></td></tr>
<tr><td>* …</td><td></td><td>+ …</td><td>+ …</td><td></td><td>(end)</td><td>+ …</td><td></td><td></td></tr>
</table>

Table 2. This seems to be the new "game of life", a family tree of prime numbers.

Will the evolution go on forever? Here is a table that shows how many partial intervals (or primes, respectively) n there are after stage s, up to s = 60:

s	p	n	s	p	n	s	p	n
1	2	**1**	21	73	**5**	41	179	**18**
2	3	**1**	22	79	**4**	42	181	**17**
3	5	**1**	23	83	**6**	43	191	**14**
4	7	**1**	24	89	**3**	44	193	**24**
5	11	**2**	25	97	**2**	45	197	**24**
6	13	**2**	26	101	**1**	46	199	**28**
7	17	**3**	27	103	**3**	47	211	**30**
8	19	**3**	28	107	**1**	48	223	**36**
9	23	**4**	29	109	**1**	49	227	**49**
10	29	**6**	30	113	**3**	50	229	**44**
11	31	**4**	31	127	**2**	51	233	**52**
12	37	**5**	32	131	**5**	52	239	**53**
13	41	**5**	33	137	**6**	53	241	**55**
14	43	**9**	34	139	**12**	54	251	**67**
15	47	**11**	35	149	**21**	55	257	**69**
16	53	**10**	36	151	**19**	56	263	**72**
17	59	**12**	37	157	**15**	57	269	**81**
18	61	**8**	38	163	**16**	58	271	**79**
19	67	**6**	39	167	**24**	59	277	**85**
20	71	**11**	40	173	**18**	60	281	**83**

Table 3. The only prime at stage 29 is 350,842,542,483,891,235,293,716,663,559,065,020,274,899,073. Talking about a bottleneck.

Although the population increases considerably after stage 33, the data doesn't provide too much confidence that it will continue to do so. We have to take a closer look at why this sequence is so erratic.

Analyzing the game

The size of the primes at each stage s with the corresponding p, p_s being the s^{th} prime, is always about $1.25*p_s\#$. The probability of a random number of this size being prime is $1/(\log p_s\# + 0.23)$, or $1/(\theta(p_s) + 0.23)$ where $\theta(p)$ is the first Chebyshev function.
For one prime q of the stage s−1 there has to be on average at least one prime in the interval $[q*p_s, (q+1)*p_s]$ in order to have a chance that the sequence keeps on producing ever more primes. More precisely, the probability of getting at least one prime out of the respective interval must be bigger than getting no prime at all.

(1) $\frac{\binom{p-1}{1}}{(\log q-1)^1}\left(1-\frac{1}{\log q}\right)^{p-1} > \frac{\binom{p-1}{0}}{(\log q-1)^0}\left(1-\frac{1}{\log q}\right)^{p-1}$ or simply $\frac{\mathrm{p}-1}{\theta(p)-0.77} > 1$

This is true for $p \geq 3$ if and only if $\theta(p) < p-0.23$, or $\theta(p) < p$ for short, since the constant value $0.23 = \log(y)$ is negligible for large p.

Now $\theta(p)$ is in fact smaller than p most of the time – but only slightly so. And indeed $p-\theta(p)$ behaves just like the mercurial $\mathrm{Li}(p)-\pi(p)$, along with the infinitude of sign changes (see Littlewood [4]):

Using the simple sum $\sum_{x=2}^{p}\frac{1}{\log x}$ for Li(p) [1)],

(2) $\mathrm{p}-\theta(p) = \log(p)\,[\mathrm{Li}(p)-\pi(p)] + 1 - \sum_{x=2}^{p-1}\left[\log(\frac{x+1}{x})(\mathrm{Li}(x)-\pi(x))\right]$

This formula makes it also clear that $p-\theta(p)$ changes sign quite a while before $\mathrm{Li}(p)-\pi(p)$ acquires negative values. For the sake of a ballpark figure: assuming $\mathrm{Li}(x)-\pi(x)$ is on average close enough to $x^{1/2}/\log(x)$, then $p-\theta(p)$ drops below zero by the time $\mathrm{Li}(p)-\pi(p) < [2p^{1/2}+O(p^{1/3})] / [(\log p)(-2+\log p)]$. It can be expected that this happens for the first time close to the first $\mathrm{Li}(p)-\pi(p)$ crossover near $1.4*10^{316}$ (Bays and Hudson [1], more extensive calculations by Saouter and Demichel [9]). The results of Platt and Trudgian [6] also confirm this.

1) As opposed to the "European convention" $\int_2^p \frac{dx}{\log x}$ and the "American convention" $\int_0^p \frac{dx}{\log x}$ which each differ from said sum by less than 1 for $p \geq 12.00501071$ (EC) or $p \geq 2$ (AC). (Actually, the sum becomes equivalent to $\int_a^p \frac{dx}{\log x}$ for a = 1.549831773645… for large p.)

Empirical data and heuristical reasoning suggests that $p-\theta(p)$ can usually be found in the vicinity of $p^{1/2}$. The bias is given by Riemann's prime counting formula [8]

(3) $\pi(x) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left[\mathrm{Li}(\sqrt[n]{x}) - \sum_\rho \mathrm{Li}(\sqrt[n]{x^\rho}) + \int_{\sqrt[n]{x}}^{\infty}\frac{du}{u(u^2-1)\log u} - \log(2)\right]$

If the Riemann Hypothesis is true, $\mathrm{Li}(p)-\pi(p)$ oscillates along $p^{1/2}/\log(p)$ with a magnitude of at most $O(p^{1/2}\log p)$ (von Koch [10]), and so $p-\theta(p)$ oscillates along $p^{1/2}$ with a magnitude of at most $O(p^{1/2}\log^2 p)$. Ingham [3] also proved this bias for sufficiently large p.

On a large scale, the number of intervals should increase slowly and asymptotically with $p-\theta(p)$, thus it may also decrease whenever $\theta(p) > p$. Yet the actual development strongly depends on the local irregularities of the distribution of prime numbers, so one can only assign the probability that the initial conditions are adequate.

Admittedly, (1) wasn't quite accurate. The number of primes in an interval after a semiprime as it is the case, especially in this short type of interval, is on average a bit smaller than for a randomly chosen interval. The following table demonstrates this sort of difference for a small interval of ten numbers:

number		divisible by						not divisible	interval sum
		2	3	5	7	11			
random +[1..10]		*50%*	*33%*	*20%*	*14%*	*9%*		*21%*	***2.08***
prime+1		100%	50%	25%	17%	10%		0%	0
prime+2		0%	50%	25%	17%	10%		28%	0.28
prime+3		100%	0%	25%	17%	10%		0%	0.28
prime+4		0%	50%	25%	17%	10%		28%	0.56
prime+5		100%	50%	0%	17%	10%		0%	0.56
prime+6		0%	0%	25%	17%	10%		56%	1.13
prime+7		100%	50%	25%	0%	10%		0%	1.13
prime+8		0%	50%	25%	17%	10%		28%	1.41
prime+9		100%	0%	25%	17%	10%		0%	1.41
prime+10		0%	50%	0%	17%	10%		38%	**1.78**

Table 4. For a random interval start and interval length 10, the average of numbers that are coprime to 2,310 = 11# is above 2, whereas for an interval of the same length after a prime number (or any number coprime to 11#), significantly less than 2 such numbers are to be expected. Hardy and Littlewood [2] laid a solid foundation regarding this issue.

Ultimately, the number of numbers that are relatively prime to k# in the intervals in question [q*p+1, (q+1)*p−1] – denoted by Φ(p−1, k) – is

$$(4)\quad \prod_{u=3}^{k}\left(1-\frac{1}{u-1}\right)\left[\left\lfloor\frac{p-1}{2}\right\rfloor+\sum_{v=3}^{\min(\frac{p-1}{2},k)}\frac{\left\lfloor\frac{p-1}{2v}\right\rfloor}{v-2}+\sum_{v_1=3}^{\min(\sqrt{\frac{p-1}{2}},k)}\sum_{v_2=v_1+2}^{\min(\frac{p-1}{2v_1},k)}\frac{\left\lfloor\frac{p-1}{2v_1v_2}\right\rfloor}{(v_1-2)(v_2-2)}+\cdots\right]$$

where the variables u and v run through prime values only. Yet disregarding the fact that p itself doesn't appear as a factor in said interval, this particular formula is only valid for 2<k<p when used as described above.

In contrast, for a random interval this is (p−1)*W(k), where $W(k) = \prod_{u=2,\ u\ prime}^{k}\left(1-\frac{1}{u}\right)$

If Φ(p−1, k) is then divided by W(k)*[(p−2)/(p−1)], then the result is a "corrected" interval length. For this, k must be appropriately large to attain the desired value (k > log(p), say).

There is a special connection between said corrected interval length and the twin prime constant C_2=0.6601618…: for k → ∞, the corrected interval length Φ(p−1, k) / W(k), on the basis of a prime number preceding the interval, can be expressed as C_2*(p−1+α), with α being a rational number defined by

(5) $2\sum_{x=1}^{\frac{p-1}{2}}\left(\prod_r \frac{r-1}{r-2} - 1\right)$

Where r runs through every distinct odd prime factor of x.

Some values of α include

p−1	α	p−1	α
2	0	20	116/15
4	0	30	6,866/495
6	2	40	141,274/8,415
8	2	50	1,329,632/58,905
10	8/3	100	129,132,288,244/2,731,483,755
12	14/3	150	123,443,421,975,532,168/1,666,745,013,838,905

Table 5. The value of α/(p−1) is asymptotic to $1/C_2-1-O(\log(p)/2pC_2)$.

While k → ∞, the "corrected" interval length (for our case with the semiprimes) is – apart from p = 3, 7, and 13 – always a bit smaller than p−1. Dividing it again by log(q*p) and we arrive at the expected average number of primes in one interval.
The ratio of the "corrected" interval length vs. p−1, which is equivalent to $\lim_{k\to\infty} \Phi(p-1, k) / [W(k)*(p-2)]$, will be denoted below by ψ(p).

Proposing practical predictions

Using these heuristics, which are so far in very good agreement with the factual data, we can start to calculate the probabilities for the game to continue.
For example, there are 594 primes at stage 100, where p = 541 and every prime $q = \lfloor p\#*Y \rfloor$ has 220 decimal digits. Each q then has a certain chance to yield n primes in the following stages:

s	p	n = 0	n = 1	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7	n > 7
101	547	34.52%	36.75%	19.53%	6.90%	1.83%	0.39%	0.07%	0.01%	0.00%
102	557	49.78%	19.46%	14.17%	8.22%	4.33%	2.16%	1.03%	0.47%	0.36%
103	563	58.57%	12.14%	10.10%	7.05%	4.63%	2.95%	1.83%	1.12%	1.62%
104	569	64.34%	8.33%	7.46%	5.75%	4.22%	3.03%	2.14%	1.50%	3.22%
105	571	68.45%	6.11%	5.72%	4.68%	3.68%	2.84%	2.17%	1.64%	4.71%
…	…	…	…	…	…	…	…	…	…	…
110	601	78.69%	2.06%	2.14%	1.99%	1.80%	1.62%	1.44%	1.27%	8.99%
150	863	90.06%	0.10%	0.11%	0.11%	0.11%	0.11%	0.11%	0.11%	9.20%
200	1,223	91.20%	0.01%	0.02%	0.02%	0.02%	0.02%	0.02%	0.02%	8.69%
1,000	7,919	91.48%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	8.52%
2,000	17,389	91.48%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	8.52%

Table 6. Only the fittest will survive.
The column for n = 0 is of importance to find out about the overall probability. Within the first few thousand values of s, this value approaches ω ≈ 91.476%. This gives the overall probability of $\omega^{594} \approx 1.039*10^{-23}$ that the sequence fails at this point.

Generally the value $\omega(s)$ can be defined for every $s \geq 1$ as the probability that a prime q at stage s will not cause an infinite sequence of primes in the further process.
For convenience, a recursive formula can be considered where, with
$\psi(p) = \lim_{k \to \infty} \Phi(p-1, k) / [W(k)*(p-2)]$ from above,

$$(6) \quad \omega(s-1) = \left(1 - \frac{\psi(p)}{\log q}\right)^{(p-1)} \sum_{j=0}^{p-1} \frac{\binom{p-1}{j}\omega(s)^j}{\left(\frac{\log q}{\psi(p)}-1\right)^j}$$

and a starting point $\omega(t) = 1-2p_t^{-1/2}$ for some t can be set to quickly compute the values for all $s < t$. Varying $\omega(t)$ between 0 and $1-\varepsilon$ for an $\varepsilon > 0$ leads to a lower and upper bound on $\omega(s)$. It may be useful that if $\omega(t) = \omega(s)+\delta$ then $\omega(t+1)$ is asymptotic to $\omega(s+1)+\delta*(1-\omega(s+1))/2$.

Again, we should bear in mind that these primes are not as flexible as (6) takes them to be: the probability of obtaining n primes in the following step is zero when n exceeds the maximal possible number of primes in the given interval (e.g. $n > 97$ for $p-1 = 546$) as fixed in the specific rules for the k-tuple conjecture. To take account of that – at least to some degree –, W(k) is deployed again such that $k\# < p$. For p = 547, this means k = 7 and W(k) = 48/210 = 1/4.375. As p gets larger than 11# = 2,310, W(k) can be adjusted to 480/2,310 = 1/4.8125 and so on.

$$(7) \quad \omega(s-1) = \left(1 - \frac{\psi(p)}{W(k)\log q}\right)^{W(k)\,(p-1)} \sum_{j=0}^{W(k)\,(p-1)} \frac{\binom{W(k)\,(p-1)}{j}\omega(s)^j}{\left(\frac{W(k)\log q}{\psi(p)}-1\right)^j}$$

This refined calculation reveals a slightly smaller value for $\omega(100)$, namely 91.435%. The entire probability that the sequence fails at this point drops by 24% to $7.937*10^{-24}$.
To show how vague these percentages are (especially at this rather early stage of computation), the actual portion of primes that don't survive after stage 100 – at least 547 out of 594 – is ≥ 92.088%; to the 594th power, this is over 68 times more than the value from the "refined" calculation! In other terms, 47 surviving primes as opposed to a predicted number of 51, from this point of view the deviation is not that bad.

$\omega(100)$ with W(1): 91.4760031% (simple formula)
$\omega(100)$ with W(2): 91.4637729% (refined formula)
$\omega(100)$ with W(3): 91.4515074%
$\omega(100)$ with W(5): 91.4422850%
$\omega(100)$ with W(7): 91.4345844%
$\omega(100)$ with W(11): 91.4291857%
It is difficult to pin down exact values there, but the refined calculation should at least give some indication on how large the error might be.

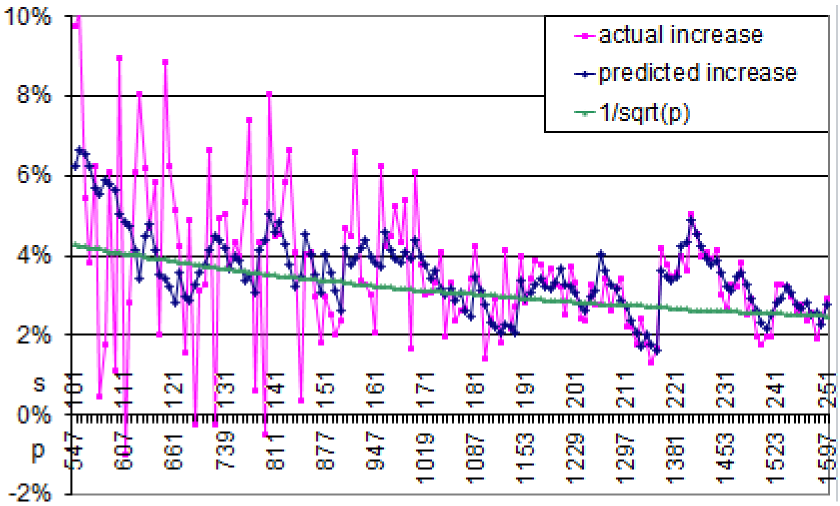
Not too surprisingly, $\omega(s)$ is largely situated in the neighborhood of $1-2p_s^{-1/2}$. This can be verified heuristically as follows:
Proceed as to be seen in Table 6, starting with stage s. Consider regular intervals with evenly distributed primes q, the size of the intervals being z*log(q) for a chosen $z \geq 1$, so for $\lim q \to \infty$, the probability that one interval doesn't contain a prime at stage s+1 is e^{-z}.
The probability to get 0 primes after t steps at stage s+t is f(s+t), where f(s)=0 and recursively $f(x+1) = e^{z[f(x)-1]}$. Then $\omega(s) = [x: e^{z(x-1)} = x]$ (= $[x: e^{zx} = e^z x]$). As $\lim z \to 1$, $x = 1-2(z-1)+O((z-1)^2)$, which can be approximated by $1-2p^{-1/2}$ for $z = p/\theta(p) \approx 1+p^{-1/2}$ (at least for small p).

What is more, for $\theta(p) = p+m*p^{1/2}$, $m = -1+O(\log^2 p)$ (Riemann Hypothesis), it can be shown that

$$(8)\quad \omega(s+1)-\omega(s) \sim (\omega(s)-1)\left[\frac{m}{\sqrt{p}}+\frac{1}{p}+\frac{1-\omega(s)}{2}+O\left(\frac{\log p}{\sqrt{p^3}}\right)+O\left(\frac{1-\omega(s)}{\sqrt{p}}\right)\right]$$

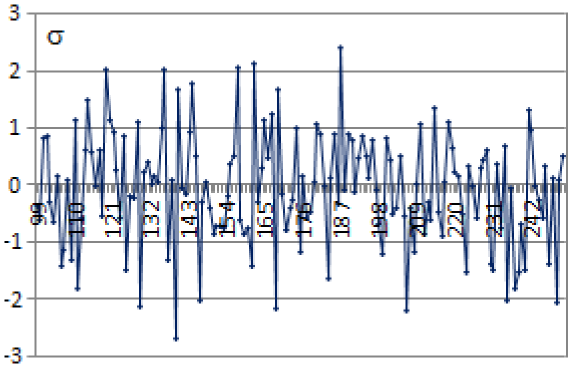
with error terms depending on either p or ω(s). As long as $m \approx -1$ as is expected on average in the long run, if ω(s) is chosen a bit larger than $1-2p_s^{-1/2}$, the values for the following ω(s+x) by the recursion formula above quickly head towards 1, so the first term in the parenthesis above, $m/p^{1/2}$, becomes the most significant. If then at some point Li(p) ≈ π(p), thus m ≈ 0, while ω(s) is very close to one, then the second term 1/p becomes the most significant, but doesn't nearly have the (then opposite) impact on ω(s) as in the region where m ≈ −1. Now what does that say? If ω(s) stays close to 1, the initially chosen value is ≥ the actual value, which is good because the actual value – the probability that the sequence terminates – is desired to be as small as possible.
An m-value of +1 would adequately counter the effect of the expected value −1, then again that should happen just as often as an m-value of −3. It should be noted that there is yet no known effective region where |m+1| ≥ 2, making it a candidate for a future endeavor.

Once the number of primes n – the number of possibilities for y – reaches a "stable" value, one may give a rough estimate as to how the sequence continues. For instance, with p = 1,153 we have n = 19,690. In the next stage, with p = 1,163, one is likely to find approximately n = 19,690*ψ(p)*(p−1)/(θ(p)+0.23) ≈ 20,298 primes.
Most of the time, the actual value lies between $n-n^{1/2}$ and $n+n^{1/2}$, and, by standard deviation measures, more than 99.7% of the time it should be well within $n\pm3n^{1/2}$. But even considering a negative deviation of $4n^{1/2}$, $-4*20{,}298^{1/2} = -569.88\ldots$ is still outnumbered by the predicted growth 20,298−19,690 = 608, corroborating the fact that the initial assumption holds with a very high probability if the distribution is normal (a sufficient but not particularly necessary condition).



Graph 1. Comparison. With the prime number output becoming more generous, the forecast is increasingly accurate.

Expanding this supposition, we can say the sequence is "stable" when n > p, where the average absolute growth rate $n/p^{1/2}$ is bigger than the majority of the local deviations of order $n^{1/2}$. This critical boundary is firmly overstepped at s = 98: p = 521 vs. n = 559 (compared to s = 97 with p = 509 vs. n = 490), and the following 153 stages seem to behave pretty "normal":



Graph 2. Standard deviation from the predicted number of primes at each stage s = 99…251. 70.6% of the values are within 1σ, 92.2% within 2σ, and all within 3σ. There are 78 negative deviations vs. 75 positive ones.

Long-term predictions suggest that the number of primes may surpass 10^6 at ≈ stage 337 (p = 2,269), 10^9 at ≈ stage 700 (p = 5,279), 10^{12} at ≈ stage 1,207 (p = 9,787), 10^{100} at ≈ stage 118,945 (p = 1,568,263), 10^{1000} at ≈ stage 19,240,445 (p = 358,604,317), and may have reached n ≈ 10^10^{155} by the time that θ(p) > p where we expect to see the next slight decline of the sequence ever since stage 139 (see Table 7).

The following approximation formula for n can be used in the medium term:

(9) $$\frac{\sqrt{e^{\sqrt{3s}}}}{8}$$

which is surprisingly accurate in spite of its simplicity in the narrow sense that it is off by less than 20, with one little exception, for s ≤ 61, off by at most 20% for 98 ≤ s ≤ 114, and off by at most 10% for 115 ≤ s ≤ 251 – compared to more elaborate predictions, the latter may hold even for 115 ≤ s ≤ 400 and beyond. Furthermore, (9) may be off by a factor of no more than 2 for 66 ≤ s ≤ 1,000+. In light of the time it would take to meticulously calculate n for all s ≤ 1,000, the exact crossover on the large side of the inequality will always remain a mystery, thus calling into question any more daring predictions regarding (9).

The status quo, pt. I: risk of failure

At stage 251 (p = 1,597) there are 117,842 probable primes, and ω(251) ≈ 95.460921%. $0.95460921^{117,842}$ ≈ **$4.08*10^{-2,378}$** – that is, for the time being, the probability that the game fails for some s > 251.

Up to that point, 4,007,583 numbers have not yet been tested for certified primality, the smallest of those being in the ballpark of $2.38*10^{247}$.
No counterexample is known for the combination of a Baillie-Pomerance-Selfridge-Wagstaff pseudoprimality test and a strong Lucas test which were performed on the probable primes, yet if there is one among all the established PRPs, it could take effect on at most 11,939 PRPs (the most prolific branch at p=601). $0.95460921^{117,842-11,939} \approx 3.97*10^{-2,137}$, i.e. $7.28*10^{240}$ times larger. It is yet unclear how much this will affect the above evaluation, especially by taking a look at Pomerance [7] who gives a heuristic argument that the number of counterexamples for a BPSW primality test up to x is >> $x^{1-\varepsilon}$ for any $\varepsilon > 0$ and sufficiently large x. In either case, said ratio can be systematically reduced by checking the primality of the prolific branches in the sequence.

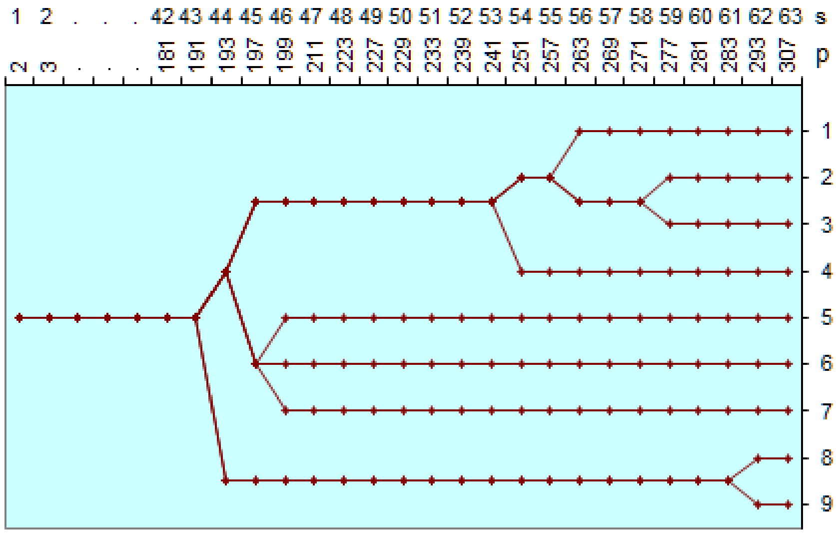
The status quo, pt. II: closing in on y

Now, what about the initially proposed values for y? The lower and upper bounds are easily obtained by picking the smallest prime $q_{s,1}$ and the largest prime $q_{s,n}$ of the last computed stage and dividing by p#: $y_{min} = q_{s,1}/p\#$ and $y_{max} = (q_{s,n}+1)/p\#$.
The calculations so far demonstrate that any y lies in a very small range between
1.25419610157801193627767955491421342377986921804262219583272255460886469942875
and
1.25419610157801193627767955491421342377986921804262219583272255460886469942905.

More precisely, y_{min} =
1.25419610157801193627767955491421342377986921804262219583272255460886469942
875144751323169673647331200713029313835829410519…
and $y_{max} = y_{min} + 2.9314187260027917698243059721184867955612840891\ldots*10^{-76}$.

Although it is not dead certain that such a y exists, if it actually does, then there should be infinitely many of those. Otherwise, we'd have to assume a unique solution where the probability of getting one prime for p → ∞ – the column for n = 1 in Table 6 – is greater than zero. But if this is the case, then at every stage a factor of approximately 1/e falls back on the column for n = 0 which then would converge to 1, challenging our assumption that there is a unique solution. It would be nice, however, to have a rigorous proof.

Having calculated all primes up to stage s, we can trace back these primes for previous stages and see which of those have "survived" in the long run. As it turns out, all probable primes from stage 251 originated from one prime at stage 43. Recall that there were a total of 14 primes at stage 43, so the other 13 "petered out" along the way, including one branch that originated in stage 37 tenaciously keeping up until it succumbed at stage 95.
The first split then appears to be at stage 44. From there on, at least two values of y may satisfy the hitherto harsh conditions. And suddenly there's a lot more to come:



Graph 3. Number of possible values for y (for all we know). Ever since s = 95, all primes emerge from one single prime at s = 43, namely
1,292,942,159,746,921,794,791,923,187,781,692,727, \
375,711,831,825,607,985,864,285,936,838,920,812,561.
Naturally, the branches in the picture are not equally strong. For s = 251 (p = 1,597), the following number of probable primes divide up on the nine given groups: 1,897 + 1,307 + 10,471 + 17,846 + 19,862 + 14,277 + 1,841 + 10,942 + 39,399.

If one was to specify these values as y_0, y_1 and so on in order to attain fixed constants for the various possibilities, one might set y_0 as the minimum value y_{min} as described above, and label each of the consecutive numbers y_1, y_2 … as the other smallest possible value(s) after a precalculated split, when it is "safe enough to assume" that those branches are stable (see also Table 8).

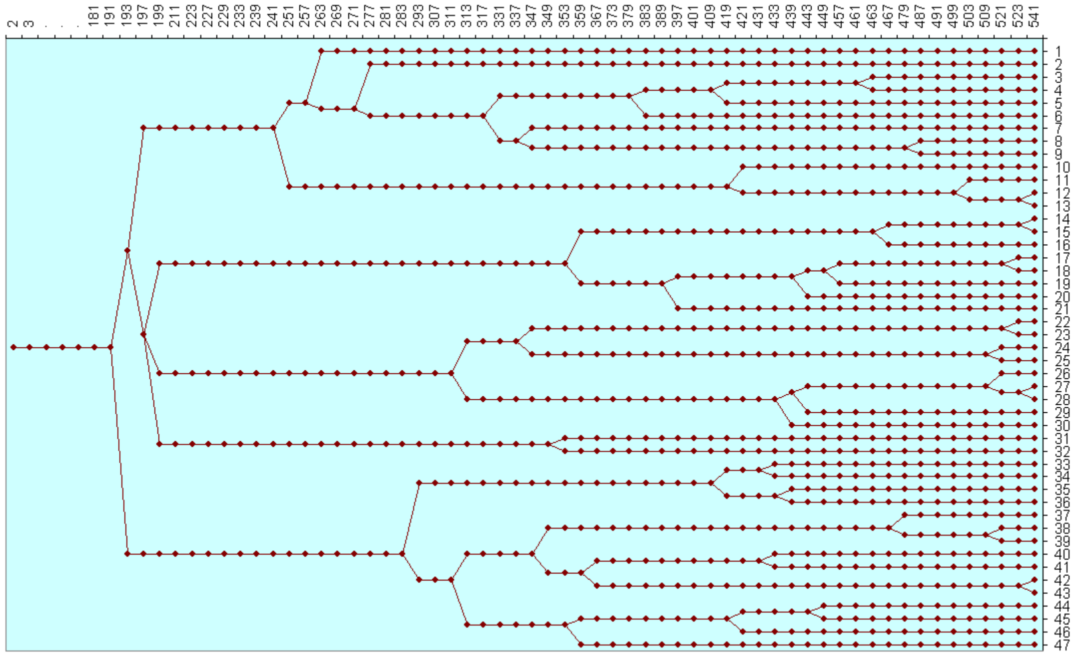
So $y_0 = y_{min}$ (see above), and then
$y_1 = y_0 + 2.9314187260027917698243\ldots*10^{-76}$
(from the first split at stage 44, $\lfloor y_0*191\#\rfloor*193+30/+88$, $y_1 \approx y_0+58/193\#$)
$y_2 = y_0 + 1.9278096567063412938136\ldots*10^{-79}$
$y_3 = y_2 + 1.1844960407807006213569\ldots*10^{-80}$
$y_4 = y_2 + 1.5406827193677358123185\ldots*10^{-80}$
$y_5 = y_0 + 1.5619786064199049976025\ldots*10^{-100}$
$y_6 = y_0 + 1.4764435119323593401152\ldots*10^{-104}$
$y_7 = y_6 + 1.4210514603041973809966\ldots*10^{-111}$
$y_8 = y_1 + 2.5503871104036223556538\ldots*10^{-119}$
…

The value y_6 is one that may be doubted more than y_0 ... y_5 since the strength of its interval (after the following split for y_7) is less than p: 1,307 < 1,597 (s = 251). Yet still given about 99.9999999999999999999999957% at this point, it may well succeed.

The stability criterion as described above, $\lim_{max} n/p < 1$ for finite branches, may even be strengthened to $\lim_{max} n/(½*p^{1/2}*\log p) < 1+\varepsilon$ and still hold for the entirety of the process for relatively small ε. Even if this is not true for all ε > 0 (and under certain conditions – looking

at runs of consecutive non-surviving primes of a given stage – can be shown to fail), in practice, it will be a safe bet that no-one will explicitly find a counterexample for branches with more than $½*p^{1/2}*\log p$ primes for respective values of p.
Yet there might be a bound $\lim_{max} n/(½*p^{1/2}*\log p) \leq c$ for finite branches for some constant c, and so far c > 0.7478 (7th branch/prime of stage 37 at stage 59 dissipating at stage 95). So two more questions are probably to remain unanswered: Is there such a bound, maybe with c > 1? If yes, what is the heuristically/actually largest c? If no, is there any other bound?

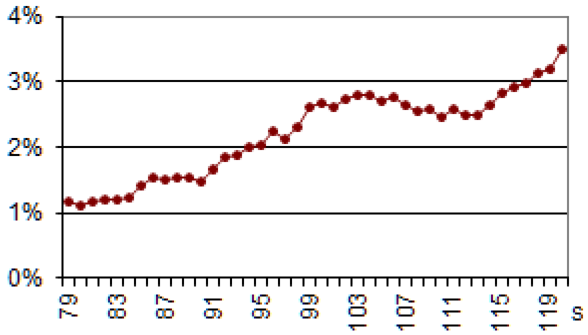
The bifurcation tree as depicted in Graph 3 gets quite impressive when expanded. For example, tracing back the primes from stage 251 to stage 100 crystallizes out 47 primes (compared to originally 594) with an eventful history.



Graph 4. Alleged number of possibilities up to stage 100.
It is interesting to note the threefold split at stage 46, p = 199. If all of these primes persist an infinite run, this still is the only point for s ≤ 100 where more than two primes out of a single prime from the previous stage will simultaneously continue the race. But how probable is this now? We have to consider the possible number of primes arising from one prime of the previous stage and the possibility that at least three of them become stable sequences, including the permutations if more than three primes show up simultaneously in the current stage:

(10) $$\sum_{x=3}^{\infty} \frac{\binom{p-1}{x}}{(\log q-1)^x}\left(1-\frac{1}{\log q}\right)^{p-1}(1-\omega_{s-1})^x\binom{x}{3}$$

For s = 101, this value is 0.0043%, thus approximately 2.76% for all 594 primes of stage 100.



Graph 5. The value is growing in the process (> 52% for s = 252), so we can soon expect another threefold split. In fact, the next one appears to be in the upper region of p = 613, s = 112.
Substituting 3 for k in the above formula, it can handily be used to calculate the approximate probability of a k-fold split for any k ≥ 2.

The possibly last stage without any split, that is without a temporarily increasing number of admissible values for y, is found at s = 103 (p = 563).

Rambling on

Instead of finding primes q for which ⌊q/p#*r#⌋ is always prime for r < p, we could go in the opposite direction and find primes for which ⌊q/r#⌋ is always prime for r# < q. Such primes include 5, 23, 83, 107, 179, 587, 1,619, and 170,099. If we allow 1 as a quasi-prime number, then this list can be extended by 2, 3, 7, 11, 47, 227, and 359. If we furthermore allow 1 < r# < q, then the composite numbers 4, (6, 10,) 14, 15, 22, (34, 35, 46,) 82, 106, 118, 119, 178, (226, 358,) 538, 539, 586, 1,618, (2,386, 2,387, 2,518, 2,519,) 12,598, 12,599, 16,738, 16,739, 17,638, 17,639, 170,098, 6,722,098, 6,722,099, 128,274,298, 128,274,299, 465,404,806, 465,404,807, (6,530,443,918, 6,530,443,919, 8,338,243,906, 8,338,243,907,) 2,804,268,786,838, 2,804,268,786,839, (13,521,051,163,486, and 13,521,051,163,487) follow. There are no others < $2.3*10^{36}$, and sample calculations suggest that there may be no others at all, the chance for only one more such number being about 0.017%.

And for those who can't get enough sequences that grow like p#*y, here is a definition for "semi-p#Y-sequences": find y such that ⌊p#*y⌋ is a prime number for every p ≥ 3. The only difference here is that ⌊2*y⌋ is allowed to be an even number other than 2 (maybe even a semiprime, depending on the taste about restrictions, though it's hard to find such an example like ⌊y⌋ = 210,457,743,323 – with 1,124 primes at s = 331). This leaves a lot of sequences that are on the verge of extinction for quite a while during the calculations. Some of them should survive an infinite run, however:

0 < y < 1: this gives a sequence which is stronger than the original p#Y-sequence
(1 < y < 2: original sequence)
25 < y < 26: stronger than the original sequence up to stage 166, weaker after stage 168
1,411 < y < 1,412: a weak sequence with only 883 primes at s=168, p=997
3,432 < y < 3,433: a very weak sequence, only 168 primes at s=168, p=997
13,948 < y < 13,949: another weak sequence

201,420 < y < 201,421: another one which has somewhat good chances to survive
6,007,103 < y < 6,007,104: rather weak
25,510,020 < y < 25,510,021: very weak, but a bit stronger than 3432 until s=167, p=991
101,422,747 < y < 101,422,748: extremely weak, only 82 primes at stage 185 (but picks up after that)

… Can you find others? Not like these, they're only finite:

2 < y < 3: extinct at stage 4
3 < y < 4: extinct at stage 7
5 < y < 6: extinct at stage 11
16 < y < 17: extinct at stage 14
978 < y < 979: extinct at stage 17
6,640 < y < 6,641: extinct at stage 23
11,456 < y < 11,457: extinct at stage 35
160,563 < y < 160,564: extinct at stage 38
283,257 < y < 283,258: extinct at stage 68
1,117,230 < y < 1,117,231: extinct at stage 78
1,594,501 < y < 1,594,502: extinct at stage 86
55,990,660 < y < 55,990,661: extinct at stage 106
108,286,142 < y < 108,286,143: extinct at stage 114

Some sketchy heuristics show that the number of y's with different integer part that result in infinite semi-sequences should also be infinite.

Additional information # 1: more on the evolution

s	p	n	s	p	n	s	p	n	s	p	n	s	p	n
61	283	**93**	97	509	**490**	133	751	**2,548**	169	1,009	**10,329**	205	1,259	**30,707**
62	293	**81**	98	521	**559**	134	757	**2,649**	170	1,013	**10,722**	206	1,277	**31,555**
63	307	**81**	99	523	**573**	135	761	**2,791**	171	1,019	**11,048**	207	1,279	**32,633**
64	311	**67**	100	541	**594**	136	769	**2,998**	172	1,021	**11,387**	208	1,283	**33,494**
65	313	**66**	101	547	**652**	137	773	**3,017**	173	1,031	**11,769**	209	1,289	**34,556**
66	317	**74**	102	557	**718**	138	787	**3,148**	174	1,033	**12,251**	210	1,291	**35,744**
67	331	**91**	103	563	**757**	139	797	**3,132**	175	1,039	**12,490**	211	1,297	**36,546**
68	337	**88**	104	569	**786**	140	809	**3,385**	176	1,049	**12,905**	212	1,301	**37,363**
69	347	**90**	105	571	**835**	141	811	**3,537**	177	1,051	**13,209**	213	1,303	**38,018**
70	349	**95**	106	577	**839**	142	821	**3,698**	178	1,061	**13,556**	214	1,307	**38,936**
71	353	**102**	107	587	**854**	143	823	**3,914**	179	1,063	**13,918**	215	1,319	**39,633**
72	359	**126**	108	593	**906**	144	827	**4,175**	180	1,069	**14,393**	216	1,321	**40,152**
73	367	**152**	109	599	**916**	145	829	**4,345**	181	1,087	**15,006**	217	1,327	**40,822**
74	373	**154**	110	601	**998**	146	839	**4,360**	182	1,091	**15,472**	218	1,361	**42,524**
75	379	**166**	111	607	**988**	147	853	**4,537**	183	1,093	**15,695**	219	1,367	**44,138**
76	383	**187**	112	613	**1,016**	148	857	**4,722**	184	1,097	**16,075**	220	1,373	**45,674**
77	389	**214**	113	617	**1,078**	149	859	**4,862**	185	1,103	**16,548**	221	1,381	**47,308**
78	397	**206**	114	619	**1,165**	150	863	**4,951**	186	1,109	**16,852**	222	1,399	**49,203**
79	401	**201**	115	631	**1,237**	151	877	**5,099**	187	1,117	**17,548**	223	1,409	**50,999**
80	409	**220**	116	641	**1,295**	152	881	**5,228**	188	1,123	**17,926**	224	1,423	**53,578**
81	419	**241**	117	643	**1,371**	153	883	**5,334**	189	1,129	**18,415**	225	1,427	**56,009**
82	421	**249**	118	647	**1,399**	154	887	**5,460**	190	1,151	**19,145**	226	1,429	**58,245**
83	431	**269**	119	653	**1,523**	155	907	**5,715**	191	1,153	**19,690**	227	1,433	**60,614**
84	433	**320**	120	659	**1,618**	156	911	**5,971**	192	1,163	**20,364**	228	1,439	**63,012**
85	439	**354**	121	661	**1,701**	157	919	**6,366**	193	1,171	**21,154**	229	1,447	**65,622**
86	443	**354**	122	673	**1,773**	158	929	**6,582**	194	1,181	**21,954**	230	1,451	**67,609**
87	449	**365**	123	677	**1,801**	159	937	**6,799**	195	1,187	**22,688**	231	1,453	**69,411**
88	457	**369**	124	683	**1,889**	160	941	**7,006**	196	1,193	**23,528**	232	1,459	**71,677**
89	461	**358**	125	691	**1,885**	161	947	**7,151**	197	1,201	**24,295**	233	1,471	**73,978**
90	463	**387**	126	701	**1,944**	162	953	**7,599**	198	1,213	**25,085**	234	1,481	**76,816**
91	467	**413**	127	709	**2,008**	163	967	**7,920**	199	1,217	**25,717**	235	1,483	**78,753**
92	479	**426**	128	719	**2,141**	164	971	**8,276**	200	1,223	**26,680**	236	1,487	**81,051**
93	487	**446**	129	727	**2,136**	165	977	**8,708**	201	1,229	**27,569**	237	1,489	**82,647**
94	491	**454**	130	733	**2,241**	166	983	**9,086**	202	1,231	**28,242**	238	1,493	**84,106**
95	499	**491**	131	739	**2,354**	167	991	**9,577**	203	1,237	**28,917**	239	1,499	**85,756**
96	503	**456**	132	743	**2,442**	168	997	**9,736**	204	1,249	**29,871**	240	1,511	**87,469**

s	p	n	s	p	n	s	p	n	s	p	n
241	1,523	**90,333**	244	1,549	**99,150**	247	1,567	**107,041**	250	1,583	**114,508**
242	1,531	**93,284**	245	1,553	**101,723**	248	1,571	**109,795**	251	1,597	**117,842**
243	1,543	**96,279**	246	1,559	**104,560**	249	1,579	**111,912**	total	**4,026,820**	

Table 7. Data for s > 60, using BPSW-pseudoprimality for s > 110. Stage 139 is the last known point where the number of primes decreases. (Interestingly enough, just before that happens, n = 4*p, which is the only known – but certainly not the only – example where n is a multiple of p.)

The calculation up to the point given above takes about a week with Pari/GP [5] on a single CPU core of a state-of-the-art PC with the following program, which in the given form starts at s = 29 and keeps track of the numbers as $a = q_{s,1}$ and a memory-friendly vector d consisting of only the consecutive differences between the numbers of a given stage where $d_1 = 0$ and, for i > 1, $d_i = q_{s,i}-q_{s,i-1}$:

```
{
a=3508425424838912352937166635590650202748990 73;
d=[0]; \\ a and d can also be read in from previously calculated data

i=0;
b=a;
while(b>1,i++;b\=prime(i));
n=#d;
gettime();
while(1,
        i++;
        p=prime(i);
        q=a*p;
        c=d;
        e=floor(exp(sqrt(3*i)/2)/7);
        d=vector(e);
        m=n;
        n=0;
        for(j=1,m,
                q+=c[j]*p;
                y=vector(p);
                forprime(b=3,p-2,
                        r=b-lift(Mod(q,b));
                        forstep(l=r,p,b,y[l]=1)
                );
                forprime(b=p+2,floor(p^1.9),
                        r=b-lift(Mod(q,b));
                        if(r<p,y[r]=1)
                );
                forstep(k=2,p-1,2,
                        if(!y[k],
                                s=q+k;
                                if(ispseudoprime(s),
                                        n++;
                                        if(n>1,d[n]=s-z,a=s);
                                        z=s
                                )
                        )
                )
        );
        g=floor(gettime()/1000);
        x="[";
        f=floor(g/3600); if(f,x=Str(x,f"h "));
        f=floor(g/60); if(f,x=Str(x,f%60"m "));
        x=Str(x,g%60"s]");
        t=Str("Level "p);
        print(t": "n" possibilities "x);
        t=Str("p#Y "t".txt");
        write(t,"a="a"; d="vecextract(d,Str("1.."n)))
)
}
```

Additional information # 2: “mille”stone

The first 1,000-digit p#Y-prime (stage 350, p = 2,357, primality proved over all stages):
29758880294466900456743298893508664189221825395080834665460679966641670340323538611062658
74760710002626916716792245132115751760492949133469320566753486211657967302638955616142480
56123980950034528686832211500014435248604833958209819278373161550816949962259892736268616
00173134044275121432942599704993068937691714687995630444495169838508853924608153197637684
34329781194956097848054398473868538461756097734062947802989545222272955432147884759489532
15376258100904758911480647631719489690751598932601226480640025811451192259365414314027908
66860125691576735599044176066331722784025970993680631190650895731667486261763755572968055
05524999457560919427667284621117997598039465130796334009757240891094211460440403405892625
06167584698451951787089619451193713787721236181896575926339065362406372104874478640876833
78420101530205735028722118337344173284720533914178728403489723916268463274475422687256573
36629787247428617722407540470581500497332256038129464337230773818385423630283496477698386
126623480539308646019

It’s a nice fact that 2,357# of all primorials has exactly 1,000 digits.

The ostensibly first 1,000-digit p#Y-prime that survives in the long run, the 16th smallest of them all at p = 2,357, spanning its branch beyond 10,499# (s > 1,284), is
29758880294466900456743298893508664189221825395080834665460679966641670340323538611062658
74760710002626916716792245132115751760492949133469320566753486211657967302638955616142480
56123980950034528686832211500014435248604833958209819278373161550816949962259892736268616
00173134044275121432942599704993068937691714687995630444495169838508853924608153197637684
34329781194956097848054398473868538461756097734062947802989545222272955432147884759489532
15376258100904758911480647631719489690751598932601226480640025811451192259365414314027908
66860125691576735599044176066331722784025970993680631190650895731667486261763755572968055
05524999457560919427667284621117997598039465130796334009757240891094211460440403405892625
06167584698451951787089619451193713787721236181896575926339065362406372104874478640883322
68449347705777327982403840592614459180457549087430477857232202558684147149935045507388563
59599138619848676656187348665352889477837618920577507694760849690878954719797386115178812
164029571622386455709,

or (1st prime of stage 350) +
64889002924617557159295368172225527028589573701517325174945374247864241568387545962282013
19902296935137242005893377980819477138898050536288244804335753007587249353108951388963748
0426037406091083077809690,

which leads to $y_{min} \geq$
1.2541961015780119362776795549142134237798692180426221958327225546088646994287514475132316
96736473312007130293138358294105190550680714644545957773479897214724735498508703837740455
38648544910432104569800625071405146132797606019221734399136669410231685026270656941987419
82202020697300428070508912125952580185561130361070261494006514520485290723407780280085443
16734128253583224021277855951903443706113250453367301336848062710640304848112964348021638
78424799778915265485948593800601469382480980040875271653561997759928697347239814789283725
10626767239907651287196033024235612690794916068309080426270552340800815890991120958620839
87783158432601041511728803871417616354122955584764825564835336501561572838431790634989265
72476414145174204072485693292701787200666665412458613015446492323282769411572687097469909
61201441174340560542916263010050038510570942549389471624556445283494302636805685513980901
74046059310097968393716013493042619454507538255565123810953362966551870774863385941160720
014897114954349144158...

Additional information # 3: tuplets (multiple primes in one interval)

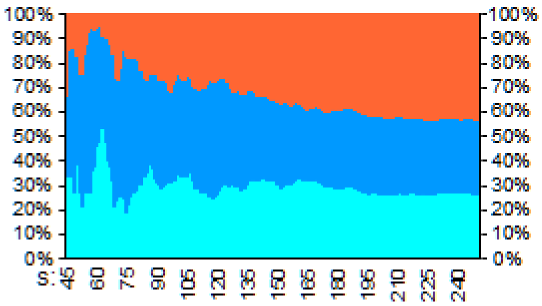
First twin: (1st prime of stage 4)*11+ {4, 10}
First triplet: (1st prime of s = 7)*19+ {10, 12, 16}
First quadruplet: (2nd prime of s = 9)*29+ {6, 14, 24, 26}
First quintuplet: (18th prime of s = 46)*211+ {42, 110, 140, 144, 194}
First sextuplet: (69th prime of s = 69)*349+ {18, 40, 234, 262, 292, 298}
First septuplet: (238th prime of s = 84)*439+ {34, 228, 252, 282, 364, 378, 382}
First octuplet: (5,687th prime of s = 159)*941+ {170, 234, 294, 462, 696, 740, 752, 812}
First nonuplet: (270th prime of s = 181)*1,091+ {18, 46, 60, 166, 180, 312, 850, 1,062, 1,080}
No decuplets up to stage 251.

The setup of the data from the Pari program above allows for a convenient search for those tuplets, setting t as the size of the tuplet, and, depending on the number of precalculated levels p_s, a starting point s and an end point e:

```
{
t=9;
s=127;
e=1597;

forprime(p=s,e,
	read(Str("p#Y Level "p".txt"));
	q=[]; f=[]; l=[]; j=0; r=0;
	for(i=1,#d,
		a+=d[i];
		if(d[i]<p,j++;r+=d[i],j=1;r=0);
		if(j==t,
			if(i==#d,d=concat(d,p+1));
			if(d[i+1]>p,
				c=vector(t);
				b=a%p-r;
				c[1]=b;
				for(k=2,t,c[k]=c[k-1]+d[i-t+k]);
				print("# "i-t+1"-"i" of "p"#: q*"p" + "c);
				q=concat(q,a\p);
				f=concat(f,c);
				l=concat(l,i)
			)
		)
	);
	if(#q,
		read(Str("p#Y Level "precprime(p-2)".txt"));
		h=0;
		i=0;
		while(h<#q,
			i++;
			a+=d[i];
			if(a==q[h+1],
				write(Str(t"-tuplets.txt"),"q("i") @ "precprime(p-
2)"# * "p" + "vecextract(f,Str(t*h+1".."t*(h+1)))" (# "l[h+1]-t+1"-
"l[h+1]")");
				h++
			)
		)
	)
)
}
```

Additional information # 4: relative strength of split intervals



Graph 6. A nice and colorful edit showing the strength of the first three split intervals, starting at stage 45. The 117,842 probable primes at stage 251 are here divided into three groups (31,521 + 35,980 + 50,341) with the respective relative strength 26.75%, 30.53%, and 42.72%. These values may or may not stabilize while the number of primes in the sequence continues to grow – proving whichever event occurs should pose an interesting task. In any case, giving the exact percentages for p → ∞ will be rather difficult.

Additional information # 5: bifurcations, specifically

For numeric information on how the branches in Graph 3 and 4 evolve, refer to this table:

s	p	split *	ω_{251}^n	s	p	split *	ω_{251}^n	s	p	split *	ω_{251}^n
44	193	0 / 1	$3*10^{-1016}$	71	353	4 / 15	$6*10^{-32}$	86	443	16 / 29	$2*10^{-14}$
45	197	0 / 2	$1*10^{-726}$	72	359	2 / 16	$9*10^{-282}$	86	443	9 / 30	$6*10^{-55}$
46	199	2 / 3	$9*10^{-289}$	72	359	10 / 17	$1*10^{-30}$	87	449	10 / 31	$1*10^{-21}$
46	199	2 / 4	$7*10^{-38}$	73	367	14 / 18	$6*10^{-16}$	88	457	16 / 32	$4*10^{-115}$
54	251	0 / 5	$9*10^{-361}$	76	383	7 / 19	$7*10^{-23}$	90	463	7 / 33	$5*10^{-18}$
56	263	0 / 6	$2*10^{-238}$	78	397	16 / 20	$6*10^{-87}$	91	467	2 / 34	$7*10^{-88}$
59	277	6 / 7	$6*10^{-212}$	81	419	7 / 21	$1*10^{-45}$	92	479	8 / 35	$1*10^{-89}$
62	293	1 / 8	$1*10^{-795}$	81	419	1 / 22	$5*10^{-97}$	93	487	12 / 36	$4*10^{-12}$
65	313	3 / 9	$3*10^{-167}$	82	421	5 / 23	$2*10^{-304}$	96	503	23 / 37	$4*10^{-199}$
65	313	8 / 10	$7*10^{-418}$	82	421	10 / 24	$7*10^{-359}$	98	521	13 / 38	$5*10^{-7}$
67	331	7 / 11	$3*10^{-26}$	84	433	1 / 25	$2*10^{-99}$	98	521	9 / 39	$1*10^{-69}$
69	347	11 / 12	$5*10^{-22}$	84	433	14 / 26	$2*10^{-6}$	98	521	35 / 40	$2*10^{-63}$
69	347	3 / 13	$1*10^{-68}$	85	439	9 / 27	$2*10^{-2}$	99	523	16 / 41	$6*10^{-22}$
70	349	8 / 14	$6*10^{-229}$	85	439	22 / 28	$4*10^{-26}$	99	523	3 / 42	$8*10^{-42}$

Table 8. With this information, the specific numbers y_i can be labeled.
* The number on the left side indicates from which branch the split comes from, while the number on the right side (i) denominates the new branch.
ω^n is an indication on the probability that the sequence of the branch terminates, taken from the number of primes n from that branch at stage 251 (p = 1,597). Branch # 27 is obviously the weakest there, only 90 out of a total of 117,842 primes stem from that line. Consequently, it has been checked further, and at stage 344 (p = 2,311), 355 primes cut ω^n back to $2*10^{-7}$.

References

[1] C. Bays and R. H. Hudson, *A new bound for the smallest x with $\pi(x) > Li(x)$*, Mathematics of Computation 69:231 (2000), pp. 1285-1296

[2] G. H. Hardy and J. E. Littlewood, *Some problems of Partitio Numerorum (III): On the expression of a number as a sum of primes*, Acta Math. 44 (1922), 1-70.

[3] A. E. Ingham, *The distribution of prime numbers*, pp. 105-106, Cambridge University Press, New York (1932)

[4] J. E. Littlewood, *Sur la distribution des nombres premiers*, Comptes Rendus, 158:1869-1872 (1914)

[5] PARI/GP, The Pari Group ©2003-2021, Univ. Bordeaux, available at http://pari.math.u-bordeaux.fr

[6] D. Platt and T. Trudgian, *On the first sign change of $\theta(x)-x$*, arXiv [math.NT]: 1407.1914 (2014)

[7] C. Pomerance, *Are there counter-examples to the Baillie-PSW primality test?*, Department of Mathematics, University of Georgia, Athens, GA 30602 U.S.A. (1984)

[8] B. Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsbericht Preussische Akademische Wissenschaft (1859), pp. 671-680

[9] Y. Saouter and P. Demichel, *A sharp region where $\pi(x)-li(x)$ is positive*, Mathematics of computation, S0025-5718(10)02351-3 (2010)

[10] H. von Koch, *Sur la distribution des nombres premiers*, Acta Math. 24 (1901), pp. 159-182