Define the sequence $n_0 = 2$, $n_{i+1} = 2^{n_i} - 1$, for $i \in \mathbb{N}$.

$n_1 = 2^2 - 1 = 3$, $n_2 = 2^3 - 1 = 7$, $n_3 = 2^7 - 1 = 127$ and
$n_4 = 2^{127} - 1 = 170141183460469231731687303715884105727$, which has 39 digits.

$n_1 = 1 + n_0$, $n_2 = 1 + m_1n_0n_1$, $n_3 = 1 + m_2n_0n_1n_2$, and $n_4 = 1 + m_3n_0n_1n_2n_3$
where $m_1 = 1$, $m_2 = 3$, and

$m_3 = 3189748471324882484658554625344658 9$
$= 3^2 \times 7 \times 19 \times 43 \times 73 \times 337 \times 5419 \times 92737 \times 649657 \times 77158673929$.

This sequence for which i = 1, 2, 3 and 4 are known Mersenne primes. **Is n_5 a prime?**

ϕ is Euler's totient function. We observe:

$n_1 = 2^{n_0} - 1 = 1 + n_0$, $\phi(n_1) = n_1 - 1 = n_0$,
$n_2 = 2^{n_1} - 1 = 1 + m_1n_0n_1$, $\phi(n_2) = n_2 - 1 = n_0n_1$, $= n_1\phi(n_1)$
$n_3 = 2^{n_2} - 1 = 1 + m_2n_0n_1n_2$, $\phi(n_3) = n_3 - 1 = 3n_0n_1n_2$ $= 3n_2\phi(n_2)$ and
$n_4 = 2^{n_3} - 1 = 1 + m_3n_0n_1n_2n_3$, $\phi(n_4) = n_4 - 1 = m_3n_0n_1n_2n_3. = k_3n_3\phi(n_3)$, $k_3 = \frac{m_3}{3}$

$n_1\phi(n_1) \mid \phi(n_2)$, $n_2\phi(n_2) \mid \phi(n_3)$, $n_3\phi(n_3) \mid \phi(n_4)$.

Theorem: n_5 is a Mersenne prime.

Proposition 1: $n_5 = 2^{n_4} - 1 \equiv 1 \pmod{\prod_{s=0}^{4} n_s}$.

Proof: $n_5 = 2^{n_4} - 1 \Rightarrow n_5 - 1 = 2^{n_4} - 2 = 2(2^{m_3n_0n_1n_2n_3} - 1) \equiv 0 \pmod{\prod_{s=0}^{4} n_s}$
$2^{n_3} \equiv 1 \pmod{n_4}, \ldots, 2^{n_0} \equiv 1 \pmod{n_1}$ since each n_i, $1 \leqslant i \leqslant 4$, is a distinct prime.
$\Rightarrow n_5 - 1 = m_4n_0n_1n_2n_3n_4$ for some $m_4 \in \mathbb{N}$ and proposition 1 is true.

But, $n_5 = 2^{n_4} - 1 \Rightarrow 2^{n_4} \equiv 1 \pmod{n_5} \Rightarrow n_4 \mid \phi(n_5)$ because n_4 is a prime.
Now, $n_5 = 2^{n_4} - 1 = 1 + m_4n_0n_1n_2n_3n_4$, $\phi(n_5) = k_4n_4$ for some $k_4 \in \mathbb{N}$.

Proposition 2: $(n_4 - 1) = \phi(n_4)$ divides $(n_5 - 1)$.

Proof: $n_5 - 1 = 2(2^{n_4 - 1} - 1) = 2(2^{\phi(n_4)} - 1)$ and $n_4 - 1 = 2(2^{\phi(n_3)} - 1)$.
Let T be the integer $2^{\phi(n_3)} - 1$. Then clearly, $2^{\phi(n_3)} \equiv 1 \pmod{T}$.
So, $2T = n_4 - 1$ and $2^{\phi(n_4)} - 1 = 2^{k\phi(n_3)} - 1 \equiv 0 \pmod{T}$ for some $k \in \mathbb{N}$,
$\Rightarrow (n_4 - 1) \mid (n_5 - 1)$ and we have the result.

We now have $n_1\phi(n_1) \mid n_2\phi(n_2) \mid n_3\phi(n_3) \mid n_4\phi(n_4) \mid (n_5 - 1)$ and $n_4 \mid \phi(n_5) \Rightarrow m_3 \mid m_4$.
So $n_5 - 1 = \frac{m_4}{m_3}n_4\phi(n_4)$ and $\frac{m_4}{m_3} \in \mathbb{N}$.

Proposition 3: $\phi(n_5) = n_5 - 1$

Proof: Consider $n_6 = 2^{n_5} - 1 = 2^{1+m_4n_0n_1n_2n_3n_4} - 1 \equiv 1$ (mod. n_5, n_4, n_3, n_2, n_1, and n_0).
$2^{n_4} \equiv 1 \pmod{n_5}$, $2^{n_3} \equiv 1 \pmod{n_4}, \ldots, 2^{n_0} \equiv 1 \pmod{n_1}$.
Since n_4, n_3, n_2, n_1, and n_0 are distinct primes, and $(n_i, n_5) = 1$ for $i \in \{0, 1, 2, 3, 4\}$, we have
$n_6 = 2^{n_5} - 1 \equiv 1 \pmod{n_5n_4n_3n_2n_1n_0}$.
$\Rightarrow 2(2^{n_5 - 1} - 1) \equiv 0 \pmod{n_5n_4n_3n_2n_1n_0} \Rightarrow n_i\phi(n_i) \mid \phi(n_5)$ for $i \in \{0, 1, 2, 3, 4\}$, $[(n_i, \phi(n_i)) = 1]$.
So, $n_5 = 2^{n_4} - 1 = 1 + m_4n_0n_1n_2n_3n_4$, $\phi(n_5) = k_4m_3n_0n_1n_2n_3n_4$ for some $k_4 \in \mathbb{N}$.
We now have $n_i\phi(n_i) \mid \phi(n_5) \Rightarrow (n_i)^2 \mid n_5$ which is impossible since $n_i \mid (n_5 - 1)$.
[Examples: $\phi(n_2) = 2 \times 3$, $3 \mid (n_2 - 1)$ and $3^2 \nmid n_2$: $\phi(n_3) = 2 \times 3^2 \times 7$, $3 \mid (n_3 - 1)$ and $3^3 \nmid n_3$, etc.]
Hence, $\phi(n_5) = n_5 - 1$ and n_5 is a Mersenne prime.

n_5 has 51,217,600,457,105,052,828,189,829,037,592,592,347 digits.

This also proves the sequence: $n_0 = 2$, $n_{i+1} = 2^{n_i} - 1$, $i \in \mathbb{N}$ defines a set of Mersenne primes.