ON THE EXCEPTIONAL SET FOR THE SUM OF A PRIME AND A k-TH POWER

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Abstract. Let $k \geq 2$ be an integer, and set

$$E_k(X) = \bigl|\{\, n \leq X,\, n \neq m^k,\, n \text{ not a sum of a prime and a } k\text{-th power } \}\bigr|.$$

We prove that there exists $\delta = \delta(k) > 0$ such that $E_k(X) \ll_k X^{1-\delta}$.

§1. Introduction

According to conjectures H and L of Hardy and Littlewood [HL] every sufficiently large number is either a k-th power or a sum of a prime number and a k-th power, for $k = 2, 3$. For any integer $k \geq 2$ set

$$E_k(X) = \bigl|\{\, n \leq X,\, n \neq m^k,\, n \text{ not a sum of a prime and a } k\text{-th power } \}\bigr|,$$

so that the Hardy–Littlewood conjectures are equivalent to

$$E_k(X) \ll_k 1,$$

for $k = 2, 3$. In the general case $k \geq 2$ Davenport and Heilbronn [DH] proved that there exists a constant $c = c(k) > 0$ such that

$$E_k(X) \ll_k \frac{X}{(\log X)^c}. \tag{1.1}$$

Schwarz [Schw] has shown, for his slightly different problem involving the k-th power of a prime, that (1.1) holds for arbitary c; see also Miech [M] for $k = 2$.

In the case $k = 2$ Brünner, Perelli and Pintz [BPP] and A. I. Vinogradov [Vi] obtained, independently and using different techniques, that there exists a positive constant δ such that

$$E_2(X) \ll X^{1-\delta}.$$

In this paper we extend the above result to the general case $k \geq 2$. We obtain the following

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Theorem. *Let $k \geq 2$ be a fixed integer. There exists $\delta = \delta(k) > 0$ such that*
$$E_k(X) \ll_k X^{1-\delta}.$$
Both δ and the implied constant are effectively computable.

We remark that Vinogradov [Vi] has stated without proof the above theorem, giving only a few hints about the modifications to be made in his proof for the case $k = 2$. However, such hints apparently lead to significant difficulties which make Vinogradov's claim quite hard to substantiate when $k \geq 3$. In fact, for $k = 2$ Vinogradov treats the singular series using Dirichlet L-series and average results for the density of their zeros. The extension of such an approach to the general case $k \geq 3$ involves the use of Dedekind zeta functions associated to suitable algebraic number fields of the form $\mathbf{Q}\bigl(\sqrt[k]{n}\bigr)$, $n \leq X$. In order to carry over the proof one would need, among others, appropriate density theorems for the zeros of such Dedekind zeta functions, on average over $n \leq X$. No hints are given by Vinogradov about the treatment of the singular series.

Our proof follows the extension, due to Brünner–Perelli–Pintz [BPP], of the ideas of Vaughan [Va1] and Montgomery–Vaughan [MV2], the main difference being the treatment of the singular series. Here we use a technique due to Vaughan, see [Va2] ch. 8, coupled with estimates for character sums over polynomial values.

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§2. Notation

We shall keep our notation consistent, as far as possible, with the corresponding one in [BPP]. We introduce the following notation: $e(x) = e^{2\pi i x}$, $e_q(x) = e(x/q)$; p shall always denote a prime number, $\operatorname{cond}\chi$ the conductor of the Dirichlet character χ, $|\mathcal{A}|$ the cardinality of the set $\mathcal{A}$. $s = \sigma + it$ will denote a complex variable. $\log_2 X$ denotes $\log\log X$. X, T, Q and P will denote large positive numbers; the constants implied in the $\ll$ and O-notations depend at most on k and are effectively computable. The value of an empty sum will be 0, of an empty product 1. $\rho = \beta + i\gamma$ will denote the generic zero of an L-function. For the sake of simplicity, we put
$$I = I(X) = \Bigl[\frac{1}{2}X, X\Bigr[\cap \mathbf{N},$$
$$J = J_k(X) = \Bigl[\frac{1}{2}X^{1/k}, X^{1/k}\Bigr[\cap \mathbf{N}.$$
We set
$$S(\alpha) = \sum_{p \in I(X)} \log p \, e(\alpha p); \qquad F_k(\alpha) = \sum_{j \in J_k(X)} e(\alpha j^k),$$
so that
$$S(\alpha)F_k(\alpha) = \sum_{X/2 \leq n < 2X} r_k(X,n) e(n\alpha),$$
where
$$r_k(X,n) = \sum_{\substack{j^k + p = n \\ j \in J_k(X) \\ p \in I(X)}} \log p.$$

We further set

$$V_k(a,q) = \sum_{h \bmod q} e_q(ah^k); \qquad S(\chi,\eta) = \sum_{p\in I(X)} \chi(p)\log p\; e(p\eta);$$

$$T_\rho(\eta) = \sum_{m\in I(X)} m^{\rho-1} e(m\eta); \qquad T(\eta) = T_1(\eta).$$

$\chi_{0,q}$ denotes the principal character mod q. For the sake of brevity, we shall also write

$$\sum_{a(q)} = \sum_{a \bmod q} = \sum_{a=1}^{q}; \qquad \sum_{a(q)}^{\star} = \sum_{a \bmod q}^{\star} = \sum_{\substack{a=1\\(a,q)=1}}^{q};$$

$$\sum_{\chi(q)} = \sum_{\chi \bmod q}; \qquad \sum_{\chi(q)}^{\star} = \sum_{\substack{\chi \bmod q\\ \chi \text{ primitive}}}.$$

$\tau(\chi) = \sum_{a(q)} \chi(a) e_q(a)$ is the Gauss sum.

$$H_k(\chi,q,n) = \sum_{a(q)} \chi(a) V_k(a,q) e_q(-na); \qquad H_k(q,n) = H_k(\chi_{0,q},q,n);$$

$$L_\rho(X,n) = \sum_{\substack{j^k+m=n\\ j\in J_k(X)\\ m\in I(X)}} m^{\rho-1}; \qquad L(X,n) = L_1(X,n);$$

$$\mathfrak{S}(n,R,r) = \sum_{\substack{q\le R\\ (q,r)=1}} \frac{\mu(q)}{q\varphi(q)} H_k(q,n); \qquad \mathfrak{S}(n,R) = \mathfrak{S}(n,R,1);$$

$$T(\chi,r,n) = \frac{\tau(\overline{\chi}) H_k(\chi,r,n)}{r\varphi(r)}.$$

We use the standard arithmetic functions: $\mu(n)$ is Möbius' function, $\varphi(n)$ is Euler's function, $\omega(n) = \sum_{p|n} 1$;

$$\rho_k(d,n) = \bigl|\{\, h \bmod d, h^k \equiv n \bmod d \,\}\bigr|;$$
$$R_{s,k}(n) = \bigl|\{\, (n_1,\dots,n_s) \in \mathbf{N}^s, n_1^k + \cdots + n_s^k = n \,\}\bigr|.$$

We will denote by c_1, $c_2, \dots$, effectively computable positive constants, which may depend on k. For the sake of brevity, we shall sometimes drop the suffix k from our functions.

§3. Dissection of the unit interval

Here we follow §3 of [BPP].

Lemma 3.1. *There exist positive constants c_1 and c_2 such that $L(s,\chi) \ne 0$ whenever*

$$\sigma \ge 1 - \frac{c_1}{\log T}; \qquad |t| \le T^{4k+7}$$

for all primitive characters χ mod q, $q \le T$, with the possible exception of at most one real primitive character $\widetilde{\chi}$ mod $\widetilde{r}$. If it exists, $L(s, \widetilde{\chi})$ has exactly one zero $\widetilde{\beta}$ not satisfying the above condition, and such zero is real, simple and satisfies

$$\frac{c_2}{\widetilde{r}^{1/2}(\log \widetilde{r})^2} \le 1 - \widetilde{\beta} \le \frac{c_1}{\log T}.$$

The proof of this result is in chapter 14 of Davenport [D]. The exceptional zero, if it exists, is called Siegel zero (relative to T).

Now let $P_1 = X^{b_1}$, where b_1 is a sufficiently small positive constant. Let us choose $T = P_1$ in Lemma 3.1 and let

$$P_2 = X^{b_2}$$

where

$$b_2 = \begin{cases} b_1 & \text{if } \widetilde{r} \le P_1^{\lambda} \\ b_1\lambda & \text{otherwise,} \end{cases}$$

where $0 < \lambda = \lambda(k) < 1/2$ is a parameter which will be specified later (see Lemma 5.8). Thus, Lemma 3.1 remains true with $T = P_2$, the value $c_1\lambda$ in place of c_1, and

$$\widetilde{r} \le P_2^{\lambda}, \tag{3.1}$$

if it exists.

We define the P_2-*excluded zeros* as those zeros of the functions $L(s, \chi)$, where χ is a primitive character mod q, $q \le P_2$, lying in the region

$$\sigma \ge 1 - \frac{(4k+2)\log_2 X}{\log X}, \qquad |t| \le P_2^{4k+7}$$

excluding the Siegel zero (relative to P_2). We define accordingly the P_2-*excluded characters* as the primitive characters χ mod r, $r \le P_2$, for which $L(\rho, \chi) = 0$, ρ being a P_2-excluded zero. The P_2-*excluded moduli* are the moduli of the P_2-excluded characters.

Now let $N^-(\alpha, T, T^{4k+7})$ denote the total number of zeros of the functions $L(s, \chi)$, where χ mod q is a primitive character, $q \le T$, in the region

$$\sigma \ge \alpha; \qquad |t| \le T^{4k+7},$$

again excluding the Siegel zero relative to T. We estimate the number of P_2-excluded zeros by means of the following

Lemma 3.2. *There exist positive constants c_3 and c_4 such that*

$$N^-(\alpha, T, T^{4k+7}) \le c_3 G(T) T^{c_4(1-\alpha)}$$

where

$$G(T) = \begin{cases} 1 & \textit{if } \widetilde{\beta} \textit{ does not exist,} \\ (1-\widetilde{\beta})\log T & \textit{if } \widetilde{\beta} \textit{ exists.} \end{cases}$$

Lemma 3.2 follows immediately from Théorème 14 of Bombieri [Bo]. It readily yields

$$\begin{aligned} N^{-}\left(1-\frac{(4k+2)\log_2 X}{\log X}, P_2, P_2^{4k+7}\right) &\le c_3 G(P_2)\exp\{(4k+2)b_2c_4\log_2 X\} \\ &\le G(P_2)(\log X)^{1/3}, \end{aligned}$$

choosing b_2 sufficiently small as a function of k. Hence

$$\big|\{P_2\text{-excluded zeros}\}\big| \le G(P_2)(\log X)^{1/3}. \tag{3.2}$$

We now define the following notation, which we shall use in the sequel

$$\begin{aligned} P &= P_2 = X^b \\ G &= G(P) \\ \mathcal{E} &= \{P\text{-excluded characters}\} \\ \mathcal{E}' &= \{P\text{-excluded zeros}\} \\ \mathcal{S} &= \{\text{Siegel character (relative to } P)\} \\ \mathcal{S}' &= \{\text{Siegel zero (relative to } P)\} \\ Q &= XP^{-4k-3}. \end{aligned} \tag{3.3}$$

For $q \le P$ and $(a,q)=1$ we let $\mathfrak{M}(a,q)$ denote the major arc

$$\left[\frac{a}{q}-\frac{1}{qQ}, \frac{a}{q}+\frac{1}{qQ}\right].$$

The major arcs are non overlapping. Set

$$\mathfrak{M} = \bigcup_{q\le P} \bigcup_{\substack{a=1\\(a,q)=1}}^{q} \mathfrak{M}(a,q)$$

Moreover, let $\mathfrak{m}$ denote the minor arcs, i. e.

$$\mathfrak{m} = \left[\frac{1}{Q}, 1+\frac{1}{Q}\right] \setminus \mathfrak{M}.$$

We have

$$\begin{aligned} r(X,n) &= \int_{1/Q}^{1+1/Q} F_k(\alpha)S(\alpha)e(-n\alpha)\,d\alpha \\ &= \int_{\mathfrak{M}} F_k(\alpha)S(\alpha)e(-n\alpha)\,d\alpha + \int_{\mathfrak{m}} F_k(\alpha)S(\alpha)e(-n\alpha)\,d\alpha \\ &= r_1(X,n) + r_2(X,n), \end{aligned} \tag{3.4}$$

say. Since the sets $\mathfrak{M}$ and $\mathfrak{m}$ are even mod 1, it is clear that $r_1(X,n)$ and $r_2(X,n)$ are real.

§4. Arithmetic lemmas

We remark that, here and in the next section, our estimates are not necessarily the sharpest known ones, but they suffice for our purposes.

Lemma 4.1. *Let* $(q_1, q_2) = 1$*, and* χ_i *be characters mod* q_i*. Then*

$$H_k(\chi_1\chi_2, q_1q_2, n) = \chi_1(q_2)\chi_2(q_1)H_k(\chi_1, q_1, n)H_k(\chi_2, q_2, n).$$

Proof. It is a straightforward consequence of the chinese remainder theorem.

Lemma 4.2. $H_k(p, n) = p(\rho_k(p, n)-1)$*. If* $\mu(q) \neq 0$ *then* $|H_k(q, n)| \le q(k-1)^{\omega(q)}$*.*

Proof. The first assertion follows in the same way as Lemma 4 in [BPP]. We recall that $0 \le \rho_k(p, n) \le (k, p-1) \le k$, hence $|H_k(p, n)| \le p(k-1)$. The second assertion now follows trivially from Lemma 4.1.

Lemma 4.3. *Let* χ *mod* q *be induced by* χ^* *mod* r*. Then*

$$\tau(\chi) = \mu\left(\frac{q}{r}\right)\chi^*\left(\frac{q}{r}\right)\tau(\chi^*)$$

and

$$|\tau(\chi^*)| = r^{1/2}.$$

Proof. See Lemma 5.2 of [MV2] and (5) of §9 of [D].

Lemma 4.4. *Let* χ *mod* q *be a primitive character. Then*

$$|H_k(\chi, q, n)| \le q^{3/2}\prod_{p|q}\left(1 - \frac{\rho_k(p, n)}{p}\right) \le q^{3/2}.$$

Proof. This Lemma easily follows from Lemma 4.3, the multiplicativity of ρ_k and Lemma 5.4 of [MV2]. See also Lemma 5 of [BPP].

Lemma 4.5. *Let* $A \in \mathbf{N}$*. We have*

$$\sum_{n\le X} A^{\omega(n)} \ll X(\log X)^{A-1}.$$

Proof. The assertion follows from formulas (2.20) and (2.28) of Wilson [W] after the trivial observation that $2^{\omega(n)} \le d(n)$.

Lemma 4.6. *Let* χ *mod* r *be a primitive character. Then we have, for* $P > r$ *and* $n \in [9X/10, X]$

$$\sum_{\substack{q\le P\\ r|q}} \frac{|\tau(\overline{\chi_{0,q}\chi})H_k(\chi_{0,q}\chi, q, n)|}{q\varphi(q)} \ll (\log P)^k.$$

Proof. We put $q = lr$ and use Lemma 4.3 to estimate $|\tau(\overline{\chi_{0,q}\chi})|$. Thus

$$\begin{aligned}\sum_{\substack{q\le P\\ r|q}} \frac{|\tau(\overline{\chi_{0,q}\chi})H_k(\chi_{0,q}\chi, q, n)|}{q\varphi(q)} &= \sum_{\substack{l\le P/r\\ (l,r)=1}} \frac{r^{1/2}\mu^2(l)}{rl\varphi(rl)}|H_k(\chi_{0,q}\chi, rl, n)| \\ &= \frac{1}{r^{1/2}\varphi(r)}\sum_{\substack{l\le P/r\\ (l,r)=1}} \frac{\mu^2(l)}{l\varphi(l)}|H_k(l, n)|\cdot|H_k(\chi, r, n)|\end{aligned}$$

by Lemma 4.1. Now, by Lemmas 4.2 and 4.4, this is

$$\leq \frac{r^{3/2}}{r^{1/2}\varphi(r)} \sum_{\substack{l \leq P/r \\ (l,r)=1}} \frac{\mu^2(l)}{\varphi(l)}(k-1)^{\omega(l)} \ll (\log P)^k,$$

applying partial summation to the result of Lemma 4.5.

§5. Analytic lemmas

Lemma 5.1. *Let $(a,q)=1$. Then*

$$F_k\left(\frac{a}{q}+\eta\right) = \frac{V_k(a,q)}{q} F_k(\eta) + O(q(1+X|\eta|)).$$

If $P < q \leq Q$ and $|\eta| \leq \dfrac{1}{qQ}$ then

$$F_k\left(\frac{a}{q}+\eta\right) \ll \frac{X^{1/k}}{P^\theta}$$

for a suitable constant $\theta = \theta(k) > 0$.

Proof. The first inequality is proved as in Lemma 3 of [DH]. The second one follows directly from Weyl's inequality, see e. g. Lemma 2.4 of [Va2].

Lemma 5.2. *For any integer $s \geq ck^2 \log k$, c being a suitable absolute constant, we have*

$$\int_0^1 |F_k(\eta)|^{2s}\, d\eta \ll X^{2s/k-1}.$$

Proof. By Parseval's inequality

$$\int_0^1 |F_k(\eta)|^{2s}\, d\eta = \int_0^1 |F_k^s(\eta)|^2\, d\eta \leq \sum_{m \leq sX} R_{s,k}(m)^2.$$

Now, by Theorem 5.4 of [Va2] we have that

$$R_{s,k}(m) \ll m^{s/k-1}$$

whenever $s \geq ck^2 \log k$, and the result follows.

Lemma 5.3. *Let $u_1, \dots, u_N$ be real numbers. For any $\delta > 0$*

$$\int_{-\delta}^{\delta} \Big| \sum_{n \leq N} u_n e(n\eta) \Big|^2 d\eta \ll \delta^2 \int_{-\infty}^{\infty} \Big| \sum_{t}^{t+(2\delta)^{-1}} u_n \Big|^2 dt.$$

Proof. This is Lemma 1 of Gallagher [G].

Lemma 5.4. *Let* $0 < \delta < \dfrac{1}{Q}$. *Then*
$$\int_{-\delta}^{\delta} |F_k(\eta)|^2 \, d\eta \ll X^{2/k-1}.$$

Proof. We have
$$\int_{-\delta}^{\delta} |F_k(\eta)|^2 \, d\eta = 2\delta \sum_{j \in J_k(X)} 1 + \sum_{j_1 \in J_k(X)} \sum_{\substack{j_2 \in J_k(X) \\ j_1 \neq j_2}} \frac{e\bigl(\delta(j_2^k - j_1^k)\bigr) - e\bigl(\delta(j_1^k - j_2^k)\bigr)}{2\pi \mathrm{i}(j_2^k - j_1^k)}.$$

We remark that the j^k are spaced $\gg X^{1-1/k}$ apart, so that, by Theorem 2 of Montgomery and Vaughan [MV1], the double sum is
$$\ll \sum_{j \in J_k(X)} X^{1/k-1} \ll X^{2/k-1}.$$

The first sum is $\ll X^{1/k} Q^{-1} \ll X^{2/k-1}$, provided that b is sufficiently small as a function of k.

Lemma 5.5. *Let* $|\gamma| \le \dfrac{X}{qQ}$ *and* $\dfrac{1}{qQ} \le |\eta| \le \dfrac{1}{2}$. *Then*
$$T_\rho(\eta) \ll \frac{X^{\beta-1}}{|\eta|}.$$

Proof. This is Lemma 12 of [BPP].

Lemma 5.6. *Let* $\left(\dfrac{1}{2} + \dfrac{1}{2^k}\right) X \le n \le X$; $|\gamma| \le \dfrac{1}{k} X^{1/k}$. *Then there exists a constant* $c_5 > 0$ *such that*
$$|L_\rho(X, n)| \le c_5 \frac{X^{1/k}}{1 + |\gamma|} X^{\beta-1}.$$

Proof. The proof is essentially the same of Lemma 13 of [BPP], using Abel's inequality and Lemmas 4.2 and 4.8 of Titchmarsh [T].

Lemma 5.7. *If* $|\gamma| \le \dfrac{1}{k} X^{1/k}$ *then*
$$\int_0^1 |F_k(\eta) T_\rho(\eta)|^2 \, d\eta \ll \frac{X^{1+2/k}}{1 + |\gamma|^2} X^{2\beta-2}.$$

Proof. The proof is essentially the same of Lemma 14 of [BPP].

Lemma 5.8. *Let* s *be as in Lemma 5.2 and* $\lambda = \dfrac{2k+1}{s}$. *If* $|\gamma| \le \dfrac{X}{qQ}$ *and* $q \le P$, *we have*
$$\int_{1/qQ}^{1/2} |F_k(\eta) T_\rho(\eta)| \, d\eta \ll X^{1/k+\beta-1} P^{-\lambda}.$$

Proof. By Lemma 5.5 we have
$$\int_{1/qQ}^{1/2} |F_k(\eta) T_\rho(\eta)| \, d\eta \le \left(\int_0^1 |F_k(\eta)|^{2s} \, d\eta \right)^{1/2s} \cdot \left(\int_{1/qQ}^{1/2} |T_\rho(\eta)|^{2s/(2s-1)} \, d\eta \right)^{(2s-1)/2s}$$
$$\ll X^{1/k+\beta-1} P^{-(2k+1)/s}.$$
∎

Lemma 5.9. *Let $n \in [9X/10, X[$ and let c_7 be a constant. There exists a constant $c_6 > 0$, depending on c_7, such that for sufficiently large X we have*

$$L(X,n) - c_7 L_{\widetilde{\beta}}(X,n) \geq c_6 G X^{1/k}.$$

Proof. By Lagrange's theorem we have

$$\begin{aligned} L(X,n) - c_7 L_{\widetilde{\beta}}(X,n) &= \sum_{\substack{n=m+j^k \\ m\in I \\ j\in J_k}} (1 - c_7 m^{\widetilde{\beta}-1}) \geq c_8 X^{1/k}(1 - P^{\widetilde{\beta}-1}) \\ &\geq c_6 X^{1/k}(1-\widetilde{\beta})\log P = c_6 G X^{1/k}. \end{aligned}$$

§6. The minor arcs

We will give an upper bound for the contribution of the minor arcs in a standard way. From Bessel's inequality and the prime number theorem we get

$$\begin{aligned} \sum_{9X/10\leq n<X} r_2(X,n)^2 &\leq \int_{\mathfrak{m}} |F_k(\alpha)S(\alpha)|^2\,d\alpha \leq \int_0^1 |S(\alpha)|^2\,d\alpha \cdot \sup_{\alpha\in\mathfrak{m}} |F_k(\alpha)|^2 \\ &\ll X\log X \cdot \sup_{\alpha\in\mathfrak{m}} |F_k(\alpha)|^2. \end{aligned}$$

From the definition of $\mathfrak{m}$ and Lemma 5.1 we have

$$\sup_{\alpha\in\mathfrak{m}} |F_k(\alpha)|^2 \ll \frac{X^{2/k}}{P^{2\theta}},$$

and hence

$$\sum_{9X/10\leq n<X} r_2(X,n)^2 \ll \frac{X^{1+2/k}}{P^{2\theta}} \log X. \tag{6.1}$$

§7. The major arcs

When $\alpha \in \mathfrak{M}(a,q)$ we write $\alpha = \dfrac{a}{q} + \eta$. Since $q \leq P$, we have $(q,p) = 1$ whenever $p > P$. Hence, by orthogonality

$$e(\alpha p) = \frac{1}{\varphi(q)} \sum_{\chi(q)} \tau(\overline{\chi})\chi(ap)e(p\eta).$$

Thus

$$S(\alpha) = \frac{1}{\varphi(q)} \sum_{\chi(q)} \tau(\overline{\chi})\chi(a)S(\chi,\eta).$$

We now define $W(\chi,\eta)$ as follows

i) If $\chi = \chi_{0,q}$ then

$$W(\chi,\eta) = S(\chi_{0,q},\eta) - T(\eta);$$

ii) if χ mod q is induced by $\chi^* \in \mathcal{E} \cup \mathcal{S}$, then $\chi = \chi_{0,q}\chi^*$ and

$$W(\chi, \eta) = S(\chi_{0,q}\chi^*, \eta) + \sum_{\substack{\rho \in \mathcal{E}' \cup \mathcal{S}' \\ L(\rho, \chi^*) = 0}} T_\rho(\eta);$$

iii) in all other cases

$$W(\chi, \eta) = S(\chi, \eta).$$

Hence

$$S\left(\frac{a}{q} + \eta\right) = \frac{\mu(q)}{\varphi(q)} T(\eta) + E(a, q, \eta) + D(a, q, \eta), \tag{7.1}$$

where

$$E(a, q, \eta) = -\sum_{\substack{\chi \in \mathcal{E} \cup \mathcal{S} \\ \mathrm{cond}\ \chi \mid q}} \sum_{\substack{\rho \in \mathcal{E}' \cup \mathcal{S}' \\ L(\rho, \chi) = 0}} \frac{\chi_{0,q}\chi(a)\tau(\overline{\chi_{0,q}\chi})}{\varphi(q)} T_\rho(\eta)$$

and

$$D(a, q, \eta) = \frac{1}{\varphi(q)} \sum_{\chi(q)} \tau(\overline{\chi})\chi(a) W(\chi, \eta).$$

We will consider D as an error term, for which only a relatively crude estimate is needed, and E as a "secondary main term", which will be evaluated in order to get, at the end, a suitably small contribution.

We shall approximate $F_k(\alpha)$ by means of Lemma 5.1

$$F_k\left(\frac{a}{q} + \eta\right) = \frac{V_k(a, q)}{q} F_k(\eta) + \Delta_k(a, q, \eta) \tag{7.2}$$

where

$$\Delta_k(a, q, \eta) \ll q(1 + X|\eta|). \tag{7.3}$$

Hence substituting (7.1) and (7.2) into (3.4) we obtain

$$\begin{aligned}
r_1(X, n) &= \sum_{q \le P} \sideset{}{^\star}\sum_{a(q)} e_q(-an) \int_{-1/qQ}^{1/qQ} F_k\left(\frac{a}{q} + \eta\right) S\left(\frac{a}{q} + \eta\right) e(-n\eta)\, d\eta \\
&= \sum_{q \le P} \frac{\mu(q)}{q\varphi(q)} H_k(q, n) \int_{-1/qQ}^{1/qQ} F_k(\eta) T(\eta) e(-n\eta)\, d\eta \\
&+ \sum_{q \le P} \frac{1}{q} \sideset{}{^\star}\sum_{a(q)} V_k(a, q) e_q(-an) \int_{-1/qQ}^{1/qQ} F_k(\eta) D(a, q, \eta) e(-n\eta)\, d\eta \\
&+ \sum_{q \le P} \frac{1}{q} \sideset{}{^\star}\sum_{a(q)} V_k(a, q) e_q(-an) \int_{-1/qQ}^{1/qQ} F_k(\eta) E(a, q, \eta) e(-n\eta)\, d\eta \\
&+ \sum_{q \le P} \sideset{}{^\star}\sum_{a(q)} e_q(-an) \int_{-1/qQ}^{1/qQ} \Delta_k(a, q, \eta) S\left(\frac{a}{q} + \eta\right) e(-n\eta)\, d\eta \\
&= S_1 + S_2 + S_3 + S_4,
\end{aligned}$$

say. In the following sections we shall deal with these sums.

§8. Evaluation of S_1

We have

$$S_1 = \sum_{q\le P} \frac{\mu(q)}{q\varphi(q)} H_k(q,n) \int_0^1 F_k(\eta)T(\eta)e(-n\eta)\,d\eta$$
$$+ O\left(\sum_{q\le P} \frac{\mu^2(q)}{q\varphi(q)} |H_k(q,n)| \int_{1/qQ}^{1/2} |F_k(\eta)T(\eta)|\,d\eta\right) \tag{8.1}$$

We now estimate the size of the error term; by Lemmas 4.2, 4.5 and 5.8 it is

$$\ll \sum_{q\le P} \frac{(k-1)^{\omega(q)}}{\varphi(q)} X^{1/k}P^{-\lambda} \ll X^{1/k}P^{-\lambda}(\log X)^k. \tag{8.2}$$

By definition, we have

$$\int_0^1 F_k(\eta)T(\eta)e(-n\eta)\,d\eta = L(X,n).$$

Thus, by (8.1), (8.2) and the definition of $\mathfrak{S}$, we have

$$S_1 = \mathfrak{S}(n,P)L(X,n) + O\Big(X^{1/k}P^{-\lambda}(\log X)^k\Big). \tag{8.3}$$

§9. Estimate of S_2

Here we follow §9 of [BPP]. Lemma 5.4 and a reduction to primitive characters yield

$$\begin{aligned}
S_2 &\ll \sum_{q\le P} \frac{1}{q\varphi(q)} \sum_{\chi(q)} \big|\tau(\overline{\chi})H_k(\chi,q,n)\big| \left(\int_{-1/qQ}^{1/qQ} |F_k(\eta)|^2\,d\eta\right)^{1/2} \cdot \\
&\qquad \cdot \left(\int_{-1/qQ}^{1/qQ} |W(\chi,\eta)|^2\,d\eta\right)^{1/2} \\
&\ll X^{(2-k)/2k} \sum_{r\le P} \sum_{\substack{q\le P\\ r|q}} \frac{1}{q\varphi(q)} \sideset{}{^\star}\sum_{\chi(r)} \big|\tau(\overline{\chi_{0,q}\chi})H_k(\chi_{0,q}\chi,q,n)\big| \cdot \\
&\qquad \cdot \left(\int_{-1/qQ}^{1/qQ} |W(\chi_{0,q}\chi,\eta)|^2\,d\eta\right)^{1/2} \\
&\ll X^{(2-k)/2k} \sum_{r\le P} \sideset{}{^\star}\sum_{\chi(r)} \left(\int_{-1/rQ}^{1/rQ} |W(\chi,\eta)|^2\,d\eta\right)^{1/2} \cdot \\
&\qquad \cdot \sum_{\substack{q\le P\\ r|q}} \frac{1}{q\varphi(q)} \big|\tau(\overline{\chi_{0,q}\chi})H_k(\chi_{0,q}\chi,q,n)\big|,
\end{aligned}$$

since $q \le P$ and $p > P$ obviously imply

$$W(\chi_{0,q}\chi, \eta) = W(\chi, \eta).$$

Now, by Lemma 4.6, the inner sum is $\ll (\log P)^k$, hence

$$S_2 \ll X^{(2-k)/2k} (\log X)^k \sum_{r \le P} \sideset{}{^\star}\sum_{\chi(r)} \left(\int_{-1/rQ}^{1/rQ} |W(\chi, \eta)|^2 \, d\eta \right)^{1/2}. \tag{9.1}$$

We use the technique based on Gallagher's Lemma, see Lemma 5.3, to estimate the sum in (9.1). Setting

$$\sideset{}{^\#}\sum_{t}^{t+h} \chi(p) \log p = \begin{cases} \displaystyle\sum_{t}^{t+h} \log p - \sum_{m=t}^{t+h} 1 & \text{if } r = 1, \\ \displaystyle\sum_{t}^{t+h} \chi(p) \log p + \sum_{\substack{\rho \in \mathcal{E}' \cup \mathcal{S}' \\ L(\rho, \chi) = 0}} \sum_{m=t}^{t+h} m^{\rho - 1} & \text{otherwise,} \end{cases}$$

and arguing as in §9 of [BPP], from (9.1) we obtain

$$\begin{aligned} S_2 &\ll X^{1/k} (\log X)^k \sum_{r \le P} \sideset{}{^\star}\sum_{\chi(r)} \max_{X/4 \le t \le X} \max_{Q/2 \le h \le PQ} \frac{1}{h} \left| \sideset{}{^\#}\sum_{t}^{t+h} \chi(p) \log p \right| \\ &= X^{1/k} (\log X)^k \cdot W, \end{aligned} \tag{9.2}$$

say. Now, as in §9 of [BPP], we follow the proof of Theorem 7 of [G] and use Lemma 4.8 of [T], thus obtaining

$$W \ll \sum_{r \le P} \sideset{}{^\star}\sum_{\chi(r)} \Bigl(\sum_{\substack{\rho \notin \mathcal{E}' \cup \mathcal{S}' \\ L(\rho, \chi) = 0 \\ |\gamma| \le P^{4k+7}}} X^{\beta - 1} + \frac{X}{QP^{4k+7}} (\log X)^2 \Bigr).$$

Lemma 3.2 yields, provided that $bc_4 < 1/2$,

$$\begin{aligned} W &\ll G \int_0^{1 - (4k+2) \log_2 X / \log X} X^{(\alpha - 1)/2} \log X \, d\alpha + X^{-1/2} + P^{-1} (\log X)^2 \\ &\ll G (\log X)^{-2k-1} + P^{-1} (\log X)^2. \end{aligned} \tag{9.3}$$

Finally, from (9.2) and (9.3) we get

$$S_2 \ll \frac{X^{1/k} G}{(\log X)^{k+1}} + \frac{X^{1/k} (\log X)^{k+2}}{P}. \tag{9.4}$$

§10. Evaluation of S_3

Let

$$\mathcal{E}_1' = \{\rho \in \mathcal{E}' \cup \mathcal{S}', |\gamma| \le P^{2k+6}\}$$
$$\mathcal{E}_2' = \mathcal{E}' \setminus \mathcal{E}_1'.$$

Working as in the last section, we have

$$\begin{aligned} S_3 = &- \sum_{\chi\in\mathcal{E}\cup\mathcal{S}} \sum_{\substack{q\le P \\ \text{cond } \chi|q}} \frac{\tau(\overline{\chi_{0,q}\chi})}{q\varphi(q)} H_k(\chi_{0,q}\chi, q, n) \sum_{\substack{\rho\in\mathcal{E}_1' \\ L(\rho,\chi)=0}} \int_{-1/qQ}^{1/qQ} F_k(\eta)T_\rho(\eta)e(-n\eta)\,d\eta \\ &- \sum_{\chi\in\mathcal{E}\cup\mathcal{S}} \sum_{\substack{q\le P \\ \text{cond } \chi|q}} \frac{\tau(\overline{\chi_{0,q}\chi})}{q\varphi(q)} H_k(\chi_{0,q}\chi, q, n) \sum_{\substack{\rho\in\mathcal{E}_2' \\ L(\rho,\chi)=0}} \int_{-1/qQ}^{1/qQ} F_k(\eta)T_\rho(\eta)e(-n\eta)\,d\eta \end{aligned}$$

(10.1)

$$=S_{3,1} + S_{3,2},$$

say. We shall treat $S_{3,1}$ as a secondary main term and $S_{3,2}$ as an error term. Now

$$\begin{aligned} S_{3,1} = &- \sum_{\chi\in\mathcal{E}\cup\mathcal{S}} \sum_{\substack{q\le P \\ \text{cond } \chi|q}} \frac{\tau(\overline{\chi_{0,q}\chi})}{q\varphi(q)} H_k(\chi_{0,q}\chi, q, n) \sum_{\substack{\rho\in\mathcal{E}_1' \\ L(\rho,\chi)=0}} \int_0^1 F_k(\eta)T_\rho(\eta)e(-n\eta)\,d\eta \\ &+ O\Bigg(\sum_{\chi\in\mathcal{E}\cup\mathcal{S}} \sum_{\substack{q\le P \\ \text{cond } \chi|q}} \frac{\left|\tau(\overline{\chi_{0,q}\chi})H_k(\chi_{0,q}\chi, q, n)\right|}{q\varphi(q)} \sum_{\substack{\rho\in\mathcal{E}_1' \\ L(\rho,\chi)=0}} \int_{1/qQ}^{1/2} |F_k(\eta)T_\rho(\eta)|\,d\eta \Bigg). \end{aligned}$$

By (3.2) and Lemmas 4.6 and 5.8, the error term is

$$\ll X^{1/k}P^{-\lambda}(\log X)^{k+1}.$$

Thus, by Lemmas 4.1 and 4.3, we have, as in §10 of [BPP], eqn. 24, that

$$\begin{aligned} S_{3,1} = &- \sum_{r\le P} \sum_{\substack{\chi \bmod r \\ \chi\in\mathcal{E}\cup\mathcal{S}}} T(\chi, r, n)\mathfrak{S}\left(n, \frac{P}{r}, r\right) \sum_{\substack{\rho\in\mathcal{E}_1' \\ L(\rho,\chi)=0}} L_\rho(X, n) \\ &+ O\Big(X^{1/k}P^{-\lambda}(\log X)^{k+1}\Big), \end{aligned} \tag{10.2}$$

where $T(\chi, r, n)$ is defined in §2. To estimate $S_{3,2}$ we need (3.2) and Lemmas 4.6 and 5.7. Arguing as in (22) of §10 of [BPP] we get

$$S_{3,2} \ll \sum_{\chi\in\mathcal{E}} \sum_{\substack{q\le P \\ \text{cond } \chi|q}} \frac{|\tau(\overline{\chi_{0,q}\chi})H_k(\chi_{0,q}\chi, q, n)|}{q\varphi(q)} \sum_{\substack{\rho\in\mathcal{E}_2' \\ L(\rho,\chi)=0}} \left(\frac{1}{qQ}\int_0^1 |F_k(\eta)T_\rho(\eta)|^2\,d\eta\right)^{1/2}$$

(10.3)

$$\ll X^{1/k}P^{-4}.$$

Collecting the estimates (10.1), (10.2) and (10.3) we finally have

$$S_3 = -\sum_{r\le P}\sum_{\substack{\chi \bmod r\\ \chi\in\mathcal{E}\cup\mathcal{S}}} T(\chi,r,n)\mathfrak{S}\left(n,\frac{P}{r},r\right)\sum_{\substack{\rho\in\mathcal{E}_1'\\ L(\rho,\chi)=0}} L_\rho(X,n)$$
$$+O\Big(X^{1/k}P^{-\lambda}(\log X)^{k+1}\Big). \tag{10.4}$$

§11. Estimate of S_4

By (3.3), (7.3) and the prime number theorem, we have

$$\begin{aligned} S_4 &\ll \sum_{q\le P}\sideset{}{^\star}\sum_{a(q)}\int_{-1/qQ}^{1/qQ} q(1+|\eta|X)\left|S\left(\frac{a}{q}+\eta\right)\right| d\eta \\ &\ll \frac{X}{Q}\sum_{q\le P}\sideset{}{^\star}\sum_{a(q)}\left(\int_{-1/qQ}^{1/qQ} d\eta\right)^{1/2}\cdot\left(\int_0^1 |S(\eta)|^2\,d\eta\right)^{1/2} \\ &\ll P^{6k+6}\log X. \end{aligned}\tag{11.1}$$

We now collect the estimates (8.3), (9.4), (10.4) and (11.1), thus obtaining that

$$r_1(X,n) = \mathfrak{S}(n,P)L(X,n) - \sum_{r\le P}\sum_{\substack{\chi\in\mathcal{E}\cup\mathcal{S}\\ \chi \bmod r}} T(\chi,r,n)\mathfrak{S}\left(n,\frac{P}{r},r\right)\sum_{\substack{\rho\in\mathcal{E}_1'\\ L(\rho,\chi)=0}} L_\rho(X,n)$$
$$+O\bigg(\frac{X^{1/k}G}{(\log X)^{k+1}}+\frac{X^{1/k}(\log X)^{k+1}}{P^\lambda}\bigg), \tag{11.2}$$

provided that b is sufficiently small as a function of k.

§12. The singular series: small moduli

In this and in the next section we deal wih the singular series $\mathfrak{S}(n,P/r,r)$, where $r=1$ or r is an excluded or Siegel modulus. We use Vaughan's technique, as in §8.6 of [Va2]. In contrast to the Dirichlet L-series approach used when $k=2$ in [BPP], a peculiar feature of Vaughan's method is that the size of the resulting exceptional set depends strongly on the size of P/r. More precisely, in order to obtain an exceptional set of size $\ll X^{1-\delta_1}$ one needs to have $P/r > X^{\delta_2}$, for suitable δ_1 and δ_2. Therefore we set $P^* = P^\nu$, $\nu=\nu(k)\in]0,1[$ to be chosen later, and deal in this section with the values $r\le P^*$. In the next section we will show that a satisfactory estimate can be obtained in a more direct way for the contribution of the values $P^* < r \le P$.

Set

$$\mathcal{F} = \{1,\ r\le P^*,\ r \text{ is an excluded or Siegel modulus }\}.$$

By Lemmas 4.1 and 4.2 we have

$$\mathfrak{S}\left(n,\frac{P}{r},r\right) = \sum_{\substack{q\le P/r\\ (q,r)=1}}\frac{\mu(q)}{\varphi(q)}\prod_{p\mid q}\big(\rho_k(p,n)-1\big).$$

Define

$$A(n,q,r)=\frac{\mu(q)}{\varphi(q)}\mu\big((q,r)^2\big)\prod_{p|q}\big(\rho_k(p,n)-1\big).$$

We remark that $A(n,q,r)$ is a multiplicative function of q. By Lemma 4.3 of [Va2] we have

$$A(n,p,r)=-\frac{1}{p-1}\mu\big((p,r)^2\big)\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0\\ \chi^k=\chi_0}}\chi(n).$$

Hence

$$A(n,p,r)=\sum_{\chi\in\mathcal{A}(p)}c(\chi)\chi(n) \tag{12.1}$$

where

$$|\mathcal{A}(p)|\le k-1,\qquad |c(\chi)|\le\frac{1}{\varphi(p)}. \tag{12.2}$$

From now on we write $R=P/r$. As in §8.6 of [Va2] we approximate $\mathfrak{S}(n,R,r)$ with a segment of the product $\prod\big(1+A(n,p,r)\big)$. In order to do this, we set $\mathcal{D}=\{q\in\mathbf{N}^*,\mu(q)\neq 0, p|q\Rightarrow p\le R\}$, and begin by estimating

$$F(n,r)=\Big|\sum_{\substack{R<q\le V\\ q\in\mathcal{D}}}A(n,q,r)\Big|, \tag{12.3}$$

where $V=\exp\big((\log P)^{1+\xi}\big)$, and $\xi>0$ is a suitable real number which we shall choose later. By (12.1), (12.2) and the multiplicativity of A, we have, when $q\in\mathcal{D}$ and $\psi>0$

$$A(n,q,r)=\sum_{\chi(q)}{}^{\star}\,c(\chi)\chi(n) \tag{12.4}$$

$$|c(\chi)|\le\frac{1}{\varphi(q)} \tag{12.5}$$

$$\sum_{\chi(q)}{}^{\star}\,|c(\chi)|^{\psi}\le\frac{(k-1)^{\omega(q)}}{\varphi(q)^{\psi}}. \tag{12.6}$$

Let $Q_0=R$, $Q_j=P^j$, $j=1,\dots,[(\log P)^{\xi}]$, and set $b(\chi)=c(\chi)$ if $q\in\mathcal{D}$, $R<q\le V$, and $b(\chi)=0$ otherwise. Hence, by (12.3) and (12.4) we have

$$F(n,r)=\Big|\sum_{R<q\le V}\sum_{\chi(q)}{}^{\star}\,b(\chi)\chi(n)\Big|.$$

By Hölder's inequality we have for any $r\in\mathcal{F}$

$$\sum_{n\in[9X/10,X[}\Big|\sum_{Q_{j-1}<q\le Q_j}\sum_{\chi(q)}{}^{\star}\,b(\chi)\chi(n)\Big|$$

$$\ll X^{1/3}\Bigg(\sum_{n\in[9X/10,X[}\Big|\sum_{Q_{j-1}<q\le Q_j}\sum_{\chi(q)}{}^{\star}\,b(\chi)\chi(n)\Big|^{3/2}\Bigg)^{2/3}. \tag{12.7}$$

We now use Lemma 8.2 of [Va2], choosing $l = j$, with (12.5), (12.6) and Lemma 4.5. The right hand side of (12.7) is

$$\ll X \bigl(\log(X^j e)\bigr)^{(j^4-1)/(6j)} \Biggl(\sum_{Q_{j-1}<q\le Q_j} \sideset{}{^\star}\sum_{\chi(q)} |b(\chi)|^{2j/(2j-1)} \Biggr)^{(2j-1)/(2j)}$$

(12.8)

$$\ll X \bigl(\log(X^j e)\bigr)^{(j^4-1)/(6j)} (\log X)^2 Q_{j-1}^{-1/2j} (\log Q_j)^{(k-2)(2j-1)/(2j)}.$$

Choosing e. g. $\xi = 1/10$ and summing over $j = 1, \dots, [(\log P)^\xi]$ and $r \in \mathcal{F}$, by (12.7) and (12.8) we finally obtain

$$\sum_{n\in[9X/10,X[} \sum_{r\in\mathcal{F}} F(n,r) \ll X P^{(\nu-1)/3}.$$

This proves that $|\{n \in \bigl[\frac{9}{10}X, X\bigl[$ such that there is an $r \in \mathcal{F}$ such that $F(n,r) \ge X^{-\delta_3}\}| \ll X^{1-\delta_3}$, for a suitable $\delta_3 = \delta_3(k,\nu) > 0$. Hence for all $n \in \bigl[\frac{9}{10}X, X\bigl[$, with $\ll X^{1-\delta_3}$ exceptions,

$$F(n,r) \ll X^{-\delta_3}, \tag{12.9}$$

for all $r \in \mathcal{F}$. We now estimate

$$G(n,r) = \sum_{\substack{q>V\\ q\in\mathcal{D}}} |A(n,q,r)|.$$

We remark that by (12.1) and (12.2) $|A(n,p,r)| \le \dfrac{k-1}{\varphi(p)}$ and set $\lambda = \dfrac{1}{\log P}$; thus we obtain

$$G(n,r) \le \sum_{q\in\mathcal{D}} \Bigl(\frac{q}{V}\Bigr)^\lambda |A(n,q,1)| \le V^{-\lambda} \prod_{p\le R} \bigl(1 + p^\lambda |A(n,p,1)|\bigr)$$

$$\le V^{-\lambda} \prod_{p\le R} \Bigl(1 + \frac{k-1}{p-1} p^\lambda\Bigr). \tag{12.10}$$

But

$$\prod_{p\le R} \Bigl(1 + \frac{k-1}{p-1} p^\lambda\Bigr) \le \prod_{p\le R} \Bigl(1 + \frac{6(k-1)}{p}\Bigr) \ll (\log P)^{6(k-1)}. \tag{12.11}$$

Hence, by (12.10) and (12.11)

$$G(n,r) \ll V^{-\lambda} (\log P)^{6(k-1)} \ll \exp\bigl\{-c_{10}(\log P)^{1/10}\bigr\}. \tag{12.12}$$

Summing up, from (12.9) and (12.12) we have that for all but $\ll X^{1-\delta_3}$ integers $n \in \bigl[\frac{9}{10}X, X\bigl[$, and all $r \in \mathcal{F}$

$$\mathfrak{S}(n,R,r) = \prod_{p\le R} (1 + A(n,p,r)) + O\Bigl(\exp\{-c_{10}(\log P)^{1/10}\}\Bigr)$$

(12.13)

$$= \prod_{p\le R} \Bigl(\frac{p-\rho_k(p,n)}{p-1}\Bigr) \prod_{\substack{p\le R\\ p\mid r}} \Bigl(\frac{p-1}{p-\rho_k(p,n)}\Bigr) + O\Bigl(\exp\{-c_{10}(\log P)^{1/10}\}\Bigr).$$

We end this section observing that for all n we have

$$\mathfrak{S}(n,R,r) \ll \sum_{\substack{q\le R\\ (q,r)=1}} \frac{\mu(q)^2}{\varphi(q)} \prod_{p|q} |\rho_k(p,n)-1| \ll (\log R)^k, \tag{12.14}$$

by Lemmas 4.2 and 4.5.

§13. The contribution of large moduli

In this section we show that the contribution to (11.2) of the excluded or Siegel moduli larger than P^* can be neglected for all $n \in \left[\frac{9}{10}X, X\right[$ but an acceptable exceptional set. In order to do this, we need a sharper estimate for $T(\chi,r,n)$ than the one provided by Lemma 4.4. From Lemma 5.4 of [MV2] we obtain

$$T(\chi,r,n) = \frac{\tau(\overline{\chi})\tau(\chi)}{r\varphi(r)}\sigma(r,\overline{\chi},n), \tag{13.1}$$

where

$$\sigma(r,\chi,n) = \sum_{h \bmod r} \chi(f(h))$$

and $f(h) = h^k - n$. In order to estimate $\sigma(r,\overline{\chi},n)$ we use an adaptation of the arguments in Burgess [Bu]. We have

Lemma 13.1. *Let χ mod r be a primitive character. Then for all but $\ll Xr^{-3/8}$ integers $n \in \left[\frac{9}{10}X, X\right[$, we have*

$$\sigma(r,\chi,n) \ll r^{1-1/7(k-1)}$$

uniformly for $r \le X/100$.

Proof. First we remark that, by (9) of [Bu], σ is multiplicative, so that we may restrict our attention to the case where r is a prime power. Suppose first that $r = p$. If $p|n$ we have trivially that

$$|\sigma(p,\chi,n)| \le p. \tag{13.2}$$

If $p\nmid kn$ the conditions of Weil's Theorem are satisfied (see e. g. Theorem 2C' of Schmidt [Schm]), and we get

$$|\sigma(p,\chi,n)| \le kp^{1/2}. \tag{13.3}$$

Clearly (13.3) holds trivially also in the case $p|k$, so that it holds whenever $p\nmid n$.

Now suppose that $r = p^\alpha$ with $\alpha \ge 2$. If $p^\alpha|n$ we have trivially that

$$|\sigma(p^\alpha,\chi,n)| \le p^\alpha. \tag{13.4}$$

If $p^\alpha \nmid n$ by (5) of [Bu] we have

$$\sigma(p^\alpha,\chi,n) = p^{\alpha-\gamma} \sum_{\substack{h=1\\ f'(h)\equiv 0 \bmod p^{\alpha-\gamma}}}^{p^\gamma} \chi(f(h)),$$

provided that γ is an integer $\geq \alpha/2$. We set

$$N(p^\gamma, p^\eta) = \sum_{\substack{h=1 \\ f'(h)\equiv 0 \bmod p^\eta}}^{p^\gamma} 1,$$

for $\eta \leq \gamma$. Suppose that $p^\beta \| k$, with $\beta \geq 0$. Since $f'(h) = kh^{k-1}$, we have that $p^\eta | f'(h)$ if and only if $p^m | h$ with $m \geq \dfrac{\eta - \beta}{k-1}$, and there are at most $p^{\gamma-(\eta-\beta)/(k-1)}$ such integers h mod p^γ. Hence

$$N(p^\gamma, p^\eta) \leq p^{\gamma-(\eta-\beta)/(k-1)},$$

so that

$$|\sigma(p^\alpha, \chi, n)| \leq p^{\alpha-\gamma} N(p^\gamma, p^{\alpha-\gamma}) \leq p^{\alpha+(\beta+\gamma-\alpha)/(k-1)},$$

for $\gamma \geq \alpha/2$. Choosing $\gamma = [(\alpha+1)/2]$, and recalling that $\alpha \geq 2$, $p^\beta \| k$ we obtain

$$|\sigma(p^\alpha, \chi, n)| \leq kp^{\alpha(1-1/3(k-1))}. \tag{13.5}$$

We set

$$g(r, n) = \prod_{\substack{p^\alpha \| r \\ p^\alpha | n}} p^\alpha, \qquad h(r, n) = \prod_{\substack{p^\alpha \| r \\ p^\alpha \nmid n}} p^\alpha$$

so that $r = g(r, n)h(r, n)$, $\big(g(r, n), h(r, n)\big) = 1$ and $g(r, n) \leq (r, n)$. Summing up, from (13.2)–(13.5) we obtain that

$$\begin{aligned} |\sigma(r, \chi, n)| &= |\sigma(g(r, n), \chi, n) \cdot \sigma(h(r, n), \chi, n)| \\ &\leq k^{\omega(r)} \cdot h(r, n)^{1-1/3(k-1)} \cdot g(r, n) \\ &\leq k^{\omega(r)} r^{1-1/3(k-1)} (r, n)^{1/3(k-1)}, \end{aligned} \tag{13.6}$$

since σ is multiplicative.

We set

$$A(X, r) = |\{n \in [\tfrac{9}{10}X, X[\,, (r, n) \geq r^{1/2}\}|,$$
$$B(r) = |\{m \bmod r, (r, m) \geq r^{1/2}\}|.$$

Obviously

$$A(X, r) \ll \left(\frac{X}{r} + 1\right) B(r).$$

Now

$$B(r) \leq \sum_{\substack{d|r \\ d\geq r^{1/2}}} \frac{r}{d} \leq d(r) r^{1/2},$$

hence

$$A(X, r) \ll X r^{-3/8}, \tag{13.7}$$

and the lemma follows from (13.6), (13.7) and the estimate $k^{\omega(r)} \ll r^{\varepsilon}$.

It is now easy to estimate the contribution to (11.2) of the excluded or Siegel moduli larger than P^*. From Lemma 13.1 and (3.2) it is clear that for all but $\ll X(P^*)^{-1/4}$ integers $n \in \left[\frac{9}{10}X, X\right[$, we have that

$$\sigma(r, \overline{\chi}, n) \ll r^{1-1/7(k-1)} \tag{13.8}$$

holds for all excluded or Siegel moduli $r \in [P^*, P]$. Choosing e. g. $\nu = 4/5$, from (3.2), (12.14), (13.1) and (13.8) we finally obtain that

(13.9)

$$\sum_{r\in[P^*,P]} \sum_{\substack{\chi \bmod r\\ \chi\in\mathcal{E}\cup\mathcal{S}}} T(\chi, r, n)\mathfrak{S}\left(n, \frac{P}{r}, r\right) \sum_{\substack{\rho\in\mathcal{E}_1'\\ L(\rho,\chi)=0}} L_\rho(X, n) \ll GX^{1/k}P^{-1/20(k-1)}$$

holds for all but $\ll XP^{-1/5}$ integers $n \in \left[\frac{9}{10}X, X\right[$.

§14. Proof of the theorem

Before completing the proof, we need two more lemmas.

Lemma 14.1. *For all but $\ll X^{1-\delta_3}$ integers $n \in \left[\frac{9}{10}X, X\right[$ and all $r \in \mathcal{F}\setminus\{1\}$, we have*

$$\left|T(\chi, r, n)\mathfrak{S}\left(n, \frac{P}{r}, r\right)\right| \le c_{11} \prod_{p\le P} \frac{p-\rho_k(p,n)}{p-1} + O\Big(\exp\left(-c_{12}(\log P)^{1/10}\right)\Big),$$

where χ is a primitive character mod r.

Proof. By Lemmas 4.3 and 4.4 we have

$$|T(\chi, r, n)| \le \frac{r}{\varphi(r)} \prod_{p|r} \left(1 - \frac{\rho_k(p,n)}{p}\right) = \prod_{p|r} \left(\frac{p-\rho_k(p,n)}{p-1}\right).$$

The Lemma now follows at once from (12.13) and the estimate

$$1 \ll \prod_{P/r<p\le P} \frac{p-\rho_k(p,n)}{p-1} \ll 1.$$

Lemma 14.2.

$$\prod_{p\le P} \frac{p-\rho_k(p,n)}{p-1} \gg (\log P)^{-k}.$$

Proof. We have $0 \le \rho_k(p,n) \le (k, p-1)$. Hence

$$\prod_{p\le P} \frac{p-\rho_k(p,n)}{p-1} \gg \prod_{k<p\le P} \left(1 - \frac{k}{p}\right) \gg (\log P)^{-k}.$$

Now we can complete the proof. From (6.1) we obtain that

$$r_2(X, n) \ll X^{1/k}P^{-\theta/2} \tag{14.1}$$

for all but $\ll XP^{-\theta/2}$ integers $n \in \left[\frac{9}{10}X, X\right[$. Let $\mathcal{C}(X)$ denote the union of all the exceptional sets encountered thus far. This means that (12.13), (13.9), Lemma 14.1 and (14.1) hold for all $n \in \left[\frac{9}{10}X, X\right[\setminus \mathcal{C}(X)$. Moreover it is clear that

$$|\mathcal{C}(X)| \ll X^{1-\delta} \tag{14.2}$$

for some $\delta = \delta(k) > 0$. Hence, from (11.2), (13.9) and the choice of ν made in §13 we have that for all $n \in \left[\frac{9}{10}X, X\right[\setminus \mathcal{C}(X)$

$$\begin{aligned} r_1(X,n) = {} & \mathfrak{S}(n,P)L(X,n) - T(\widetilde{\chi},\widetilde{r},n)\mathfrak{S}\left(n, \frac{P}{\widetilde{r}}, \widetilde{r}\right) L_{\widetilde{\beta}}(X,n) \\ & - \sum_{r \le P^*} \sum_{\substack{\chi \bmod r \\ \chi \in \mathcal{E}}} T(\chi,r,n)\mathfrak{S}\left(n, \frac{P}{r}, r\right) \sum_{\substack{\rho \in \mathcal{E}_1' \setminus \mathcal{S}' \\ L(\rho,\chi)=0}} L_\rho(X,n) \\ & + O\left(\frac{X^{1/k}G}{(\log X)^{k+1}} + \frac{X^{1/k}(\log X)^{k+1}}{P^\lambda}\right), \end{aligned} \tag{14.3}$$

where λ is defined as in Lemma 5.8. Here it is tacitly understood that the term containing the Siegel zero is to be deleted whenever $\widetilde{r} > P^* = P^{4/5}$ or $\widetilde{r}$ does not exist.

From (12.13) and Lemmas 14.1 and 14.2 we obtain that

$$\mathfrak{S}(n,P) \ge \frac{1}{2} \prod_{p \le P} \frac{p - \rho_k(p,n)}{p-1} \tag{14.4}$$

and

$$\left| T(\chi,r,n)\mathfrak{S}\left(n, \frac{P}{r}, r\right) \right| \le 2c_{11} \prod_{p \le P} \frac{p - \rho_k(p,n)}{p-1} \tag{14.5}$$

for $r \in \mathcal{F} \setminus \{1\}$, if P is sufficiently large. Hence from Lemma 5.9 and (14.3)–(14.5) for all $n \in \left[\frac{9}{10}X, X\right[\setminus \mathcal{C}(X)$ we have that

$$\begin{aligned} r_1(X,n) \ge {} & \frac{1}{2} \prod_{p \le P} \frac{p - \rho_k(p,n)}{p-1} \left(c_6 G X^{1/k} - 4c_{11} \sum_{r \le P^*} \sum_{\substack{\chi \bmod r \\ \chi \in \mathcal{E}}} \sum_{\substack{\rho \in \mathcal{E}_1' \setminus \mathcal{S}' \\ L(\rho,\chi)=0}} |L_\rho(X,n)| \right) \\ & + O\left(\frac{GX^{1/k}}{(\log X)^{k+1}} + \frac{X^{1/k}(\log X)^{k+1}}{P^\lambda}\right). \end{aligned} \tag{14.6}$$

From Lemmas 3.2 and 5.6, as in (35) of [BPP], we obtain that

$$4c_{11} \sum_{r \le P^*} \sum_{\substack{\chi \bmod r \\ \chi \in \mathcal{E}}} \sum_{\substack{\rho \in \mathcal{E}_1' \setminus \mathcal{S}' \\ L(\rho,\chi)=0}} |L_\rho(X,n)| \le \frac{1}{2} c_6 G X^{1/k} + O(1) \tag{14.7}$$

provided that b is sufficiently small. But by (3.1) and Lemma 3.1, the Siegel zero $\widetilde{\beta}$, if it exists, satisfies

$$G = (1 - \widetilde{\beta}) \log P \ge \frac{c_{13}}{P^{\lambda/2} \log X}. \tag{14.8}$$

From (14.6)–(14.8) and Lemma 14.2 we obtain

$$r_1(X,n) \gg \frac{GX^{1/k}}{(\log X)^k} \tag{14.9}$$

for every $n \in \left[\frac{9}{10}X, X\right[\setminus \mathcal{C}(X)$.

By Lemmas 5.2 and 5.8 we have that $\lambda = \dfrac{2k+1}{s}$ and $s > ck^2 \log k$. Choosing s such that

$$\lambda = \frac{2k+1}{s} \leq \frac{\theta}{2},$$

from (3.4), (14.1), (14.2), (14.8) and (14.9) we finally obtain that

$$r(X,n) \gg \frac{GX^{1/k}}{(\log X)^k}$$

for all but $\ll X^{1-\delta}$ integers $n \in \left[\dfrac{9}{10}X, X\right[$, where $\delta = \delta(k) > 0$. The Theorem now follows splitting the interval $[1, X[$ into intervals of type $\left[\dfrac{9}{10}t, t\right[$.

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